Math 249B, Winter 2021: Newton-Thorne symmetric power automorphy lifting

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Abstract

Richard Taylor taught a course on Newton-Thorne symmetric power automorphy lifting at Stanford in Winter 2021.

These are scribed notes from the course. Conventions are as follows: Each lecture gets its own "chapter," and appears in the table of contents with the date.

Of course, these notes are not a faithful representation of the course, either in the mathematics itself or in the quotes, jokes, and philosophical musings; in particular, the errors are our fault. By the same token, any virtues in the notes are to be credited to the lecturer and not the scribe(s).[1] Please email suggestions to lynnelle@stanford.edu.

Contents

[1]This abstract has been adapted from Akhil Mathew's introduction to his notes, with his permission.

1 January 12: introduction.

This quarter, we are discussing Newton-Thorne's work giving holomorphic continuation (HC) of symmetric power L-functions. (We have generally found it easier to get meromorphic continuation (MC), because that follows from an L-function being automorphic after an uncontrollable finite base change ["potentially automorphic"], whereas HC requires automorphy with the original base.)

1.1 Statement of theorem

Let f be a cuspidal holomorphic eigenform of weight $k \geq 2$ (k must be even) for $SL_2(\mathbb{Z})$. Recall that f is a function

$$f : \mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{im}(\tau) > 0\} \to \mathbb{C}$$

such that

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau) \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

and also $f(\tau) \to 0$ as $\tau \to i\infty$. For all p we have

$$T_p f = \frac{1}{p} \sum_{i=0}^{p-1} f\left(\frac{\tau + i}{p}\right) + p^{k-1} f(p\tau) = a_p f$$

for some $a_p \in \mathbb{C}$. The T_ps for all p are simultaneously diagonalizable, and the $\mathbb{C}$-vector space S_k of such f is finite-dimensional. The best-known example is

$$\Delta(\tau) = e^{2\pi i\tau} \prod_{n=1}^{\infty} (1 - e^{2\pi in\tau})^{24} = e^{2\pi i\tau} - 24e^{4\pi i\tau} + \cdots = \sum_{n=1}^{\infty} \tau(n) e^{2\pi in\tau}$$

where $T_p\Delta = \tau(p)\Delta$. Factor

$$x^2 - a_p x + p^{k-1} = (x-\alpha_p)(x-\beta_p).$$

Deligne showed that $|\alpha_p| = |\beta_p| = p^{(k-1)/2}$. For $n \geq 1$, we define

$$L(f, \mathrm{Sym}^{n-1}, s) = \prod_p \prod_{i=0}^{n-1} \left(1 - \frac{\alpha_p^i \beta_p^{n-1-i}}{p^s}\right)^{-1}.$$

Because of Deligne's bounds, this is holomorphic on $\Re(s) > 1 + \frac{(n-1)(k-1)}{2}$. We define

$$L_{\infty,n,k}(s) = \prod_{i=1}^{n/2} 2(2\pi)^{-(s+(i-n/2)(k-1))}\Gamma(1 + (i-n/2)(k-1))$$

if n is even, and

$$\begin{aligned} L_{\infty,n,k}(s) =& \pi^{-\frac{1}{2}\left(s - \frac{(n-1)(k-1)}{2} + \delta\right)}\Gamma\left(\frac{s - \frac{1}{2}(n-1)(k-1) + \delta}{2}\right) \\ &\times \prod_{i=1}^{(n-1)/2} 2(2\pi)^{-\left(s + \left(i - \frac{n-1}{2}\right)(k-1)\right)}\Gamma\left(s + \left(i - \frac{n-1}{2}\right)(k-1)\right) \end{aligned}$$

if n is odd, where

$$\delta = \begin{cases} 0 & \text{if } n \equiv 1 \pmod 4 \\ 1 & \text{if } n \equiv -1 \pmod 4. \end{cases}$$

Theorem 1.1.1 (Newton-Thorne). *$L(f, \mathrm{Sym}^{n-1}, s)$ continues to a holomorphic function on all of $\mathbb{C}$ and satisfies the functional equation*

$$\begin{aligned} L_{\infty,n,k}((n-1)(k-1)+1-s)&L(f, \mathrm{Sym}^{n-1}, (n-1)(k-1)+1-s) \\ &= (-1)^{k(n-1)/2} L_{\infty,n,k}(s) L(f, \mathrm{Sym}^{n-1}, s) \end{aligned}$$

(except if $n=1$, where there is a simple pole at $s=1$).

1.2 History

For $n = 1$, we have $L(f, \mathrm{Sym}^0, s) = \zeta(s)$, and Riemann showed in 1859 that

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{(s-1)/2}\Gamma((1-s)/2)\zeta(1-s).$$

For $n = 2$, Hecke showed in 1936 that

$$(2\pi)^{-s}\Gamma(s)L(f, \mathrm{Sym}^1, s) = (-1)^{k/2}(2\pi)^{2-k}\Gamma(k-s)L(f, \mathrm{Sym}^1, k-1)$$

using $\Gamma(s) = \int_0^\infty f(iy)y^{s-1}dy$. For $n = 3$, Shimura showed in 1975 that

$$2^{-s}\pi^{-3s/2}\Gamma(s)\Gamma\left(\frac{s+2-k}{2}\right)L(f, \mathrm{Sym}^2, s)$$

is the same as this expression evaluated at $2k-1-s$. As in the previous cases, he did this by writing the LHS in the form

$$(2\pi)^{-s}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+1-k}{2}\right)\int_{\mathbb{H}/\Gamma_0(4)} f(x+iy)\overline{h(x+iy)}E_k(x+iy,s-1)\frac{dxdy}{y^2}$$

where $h(\tau)=\frac{1}{2}+\sum_{n=1}^{\infty}e^{2\pi i n^2\tau}$ and $E_k(x+iy,s-1)$ is a 1/2-integral-weight Eisenstein series, so that analytic continuation of the L-function follows from analytic continuation for the Eisenstein series. (Rankin 1939 had already established meromorphic continuation and the functional equation (FE), using a different analytic method.) $n=4,5$ was done by Kim-Shahidi 1999-2002 (again, MC+FE was done first by Langlands and Shahidi 1971). MC+FE for all n was completed in 2010; HC was done for $n=6,7,8,9$ by Clozel-Thorne 2014-17.

For $n>4$, instead of analytic techniques, the general strategy has been to show that there exists a cuspidal automorphic representation π_n on $GL_n(\mathbb{A})$ such that

$$L(f,\mathrm{Sym}^{n-1},s)=L(\pi_n,s).$$

Then $L(\pi_n,s)$ has the desired properties by Godement-Jacquet 1972. For $n=5$, Kim-Shahidi used the converse theorem of Cogdell and Piatetski-Shapiro, going through suitable twists by automorphic representations. For $n>5$, we use Galois representations.

Fix $\mathbb{C}\cong\overline{\mathbb{Q}}_l$ and let $G_{\mathbb{Q}}:=\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Deligne showed how to associate f to a representation $\Gamma_f:G_{\mathbb{Q}}\to GL_2(\overline{\mathbb{Q}}_l)$ which is continuous and unramified outside l, such that if $p\neq l$ then $\mathrm{tr}\,\Gamma_f(\mathrm{Frob}_p)=a_p$. Then taking symmetric powers gives

$$\mathrm{Sym}^{n-1}\Gamma_f:\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\to GL_n(\overline{\mathbb{Q}}_l).$$

It turns out that $L(\mathrm{Sym}^{n-1}\Gamma_f,s)=L(f,\mathrm{Sym}^{n-1},s)$. So we want to find π_n such that $L(\mathrm{Sym}^{n-1}\Gamma_f,s)=L(\pi_n,s)$. If such π_n can be found, we say that $\mathrm{Sym}^{n-1}\Gamma_f$ is automorphic.

To find such a π_n, we use the idea going back to Wiles of proving an automorphy lifting theorem. We assume that

$$\mathrm{Sym}^{n-1}(\Gamma_f \pmod{l})=\mathrm{Sym}^{n-1}(\overline{\Gamma}_f)$$

is automorphic, i.e. there is a cuspidal automorphic representation π_n' such that

$$L(\mathrm{Sym}^{n-1},\Gamma_f,s)\equiv L(\pi_n',s)\pmod{l}$$

(this doesn't make sense as stated, but what we mean is that they are congruent Euler factor by Euler factor, thought of as formal power series in p^{-s}). We prove that assuming this and other conditions, $\mathrm{Sym}^{n-1}\Gamma_f$ is automorphic. Another important step is showing potential automorphy: after some finite base change, $\mathrm{Sym}^{n-1}(\Gamma_f \pmod{l})$ is automorphic. Potential automorphy gives MC+FE but not HC.

1.3 Newton-Thorne strategy

Suppose $\overline{\Gamma}_f=\mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}\overline{\chi}$, where $K/\mathbb{Q}$ is imaginary quadratic. Then we have

$$\mathrm{Sym}^{n-1}\overline{\Gamma}_f=\bigoplus_{i=0}^{\lfloor n/2\rfloor}\mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}(\overline{\chi}^i(\overline{\chi}^c)^{n-1-i})(\oplus\overline{\epsilon}^{1-k}\text{ if } n \text{ is odd})$$

where ϵ is the cyclotomic character. The summands are known to be automorphic, because they come from Eisenstein series on GL_n associated to a parabolic subgroup with 2×2 blocks, but possibly not from a cusp form. But Clozel observed that if you descend from GL_n to $U(n)$, an Eisenstein series can descend to a cusp form that is "unstable"/"endoscopic" in the Langlands-Arthur theory of the trace formula. So you can try to do automorphy lifting with this. This is what Clozel-Thorne did, but automorphy lifting theorems frequently require the residual representation to be irreducible, so you need to prove weak versions that work when it is reducible, which is difficult, so they only did it in small cases.

Newton-Thorne's strategy is as follows. Let $\psi : G_{\mathbb{Q}(i)} \to \overline{\mathbb{Q}}_l^\times$ be unramified outside l and crystalline at l with Hodge-Tate numbers $(0, 4)$.

1. Prove that for n odd, there is a solvable CM extension $F/\mathbb{Q}$ (solvable extensions are good because you can descend through them by base change) and a regular algebraic cuspidal (RAC) automorphic representation π_n' of $GL_n(\mathbb{A}_F)$ such that

   (a) $(\pi_n')^c \cong (\pi_n')^\vee \|\det\|^{1-n}$ (we frequently abbreviate this plus RAC as PRAC, where P is for "polarized"),

   (b) $\pi_{n,v}'$ is Steinberg for some v (this is analogous to asking for an elliptic curve with multiplicative reduction, or for something of exact level $\Gamma_0(v)$),

   (c) $\overline{\Gamma_{\pi_n'}} \cong \operatorname{Sym}^{n-1}\left(\operatorname{Ind}_{G_{\mathbb{Q}(i)}}^{G_\mathbb{Q}} \overline{\psi}\right)(\epsilon^3\delta)^{(n-1)/2}$ where ϵ is the cyclotomic character and $\delta : \operatorname{Gal}(\mathbb{Q}(i)/\mathbb{Q}) \xrightarrow{\sim} \{\pm 1\}$; recall that the latter splits as
   $$\left(\operatorname{Ind}_{G_{\mathbb{Q}(i)}}^{G_\mathbb{Q}} \overline{\psi}^{n-1}\right)(\epsilon^3\delta)^{(n-1)/2} \oplus \left(\operatorname{Sym}^{n-3}\operatorname{Ind}_{G_{\mathbb{Q}(i)}}^{G_\mathbb{Q}}(\overline{\psi})\right)(\epsilon^3\delta)^{(n-3)/2}\epsilon^{-1}.$$

   The above splitting makes it possible to do induction on n: by the inductive hypothesis the second term is automorphic on GL_{n-3} (coming from something locally Steinberg), and the first term comes from a theta series on GL_2. Using that $U(2) \times U(n-2)$ is endoscopic for $U(n)$, we get an automorphic form on $U(n)$, which might not be Steinberg. Level-raising makes it Steinberg, but no longer endoscopic, hence "stable". Maintaining the Steinberg condition makes it possible to produce automorphy lifting theorems for residually reducible representations.

2. Prove the same thing for all $n = 2^m n'$ where n' is odd, this time inducting on m. This is similar: if n is even, and we we choose $\omega : G_{\mathbb{Q}(i)} \to \overline{\mathbb{Q}}_l^\times$ with $\omega\omega^c = \epsilon^{-3}\delta$ (a well-behaved character), we have
   $$\left(\operatorname{Sym}^{n-1}\operatorname{Ind}_{G_{\mathbb{Q}(i)}}^{G_\mathbb{Q}}(\overline{\psi})\right)\overline{\omega}^{n-1}$$
   $$\cong \left(\operatorname{Sym}^{n/2-1}\operatorname{Ind}_{G_{\mathbb{Q}(i)}}^{G_\mathbb{Q}} \overline{\psi}\right)\omega^{n/2-1}(\psi\omega)^{n/2} \oplus \left(\operatorname{Sym}^{n/2-1}\operatorname{Ind}_{G_{\mathbb{Q}(i)}}^{G_\mathbb{Q}} \overline{\psi}\right)\omega^{n/2-1}(\psi^c\omega)^{n/2}.$$
   Then the Eisenstein series for
   $$\pi_{n/2}' \otimes (\psi\omega)^{n/2} \boxplus \pi_{n/2}'(\psi^c\omega)^{n/2}$$
   gives an unstable/endoscopic cusp form on $U(n)$ (using that $U(n/2) \times U(n/2)$ is endoscopic for $U(n)$), and again we can raise the level to get the Steinberg condition. This step is easier than the previous one.

3. There is $q \equiv 3 \pmod 4$ and a PRAC π^q of $GL_2(\mathbb{A})$ such that

   - π^q_v is unramified outside q and 2 (it is level $4q$)
   - π^q_q is Steinberg
   - $\pi^q_2 = PS(\chi_1, \chi_2)$ where χ_1 is unramified and χ_2 has conductor 4
   - $\mathrm{Sym}^{n-1} \Gamma_{\pi^q}$ is automorphic for all n
   - $\overline{\Gamma_{\pi^q}} = \mathrm{Ind}_{G_{\mathbb{Q}(i)}}^{G_{\mathbb{Q}}} \overline{\psi}$.

   This follows from the previous two steps and base change. After these steps, we can prove the theorem for one modular form f (of level $4q$).

4. There is π a PRAC automorphic representation of $GL_2(\mathbb{A})$ unramified everywhere with $\mathrm{Sym}^{n-1} \pi$ automorphic for all n. (i.e. the same statement as before, but now f is level 1 for some weight; π^q had level $4q$.) This uses the eigencurve, which parametrizes p-adic families of automorphic forms, as follows. You prove that if a symmetric power lifting exists for one point on the eigencurve, it exists everywhere on the same irreducible component. π^q gives a point on the q-adic eigencurve, and if you specialize elsewhere on the same component, you get some π' that also has symmetric power lifts and is only ramified at 2, so is level 4. Then π' gives a point on the 2-adic eigencurve, and elsewhere on the same component you can find the desired unramified π.

   Now we have the theorem for one modular form of level 1.

5. Consider the 2-adic eigencurve of tame level 1. The geometry of this is very well-studied. Recall that the eigencurve lives in the product of the weight space $\mathscr{W}$ and $\mathbb{G}_m^{an}$ (the latter giving the U_2-eigenvalue). Let κ be the projection of the eigencurve to weight space. If f is weight k and has nebentypus χ at 2, we have $\kappa(f) = s^{k-2}\chi(s) - 1$. $\mathscr{W}$ is an open disc, and the boundary of $\mathscr{W}$ is where $8 \nmid \kappa(f)$. In this region, the eigencurve decomposes into pieces X_i where

   (a) X_i is an annulus mapping isomorphically under κ to an annulus in weight space,

   (b) on X_i we have $|a_2|_2 = |\kappa|_2^i$,

   (c) for each classical point on X_i of weight k, there is a 'twin' point on $X_{(k-1)/v_2(\kappa)-i}$.

   So given $x \in X_i$, we can get a new point at weight $2i + 2^{m+1} - 1$ for $m \gg 0$; then take its twin on X_{2^m-1}; then get a point over $\kappa = \zeta_{2^{m+1}} - 1$ (weight 2, highly ramified); then take its twin on X_1. The point is that for $p = 2$ and tame level 1, the image of the eigencurve in the universal Galois space is connected via these two operations. So we can move existence of symmetric powers from one form to any other.

Remark 1. In subsequent work, Newton-Thorne removed the level-1 assumption from the theorem.

1.4 Course plan

We will go through as many of the following topics as possible in order.

1. l-adic representations (no proofs).
2. Automorphic forms on GL_n and unitary groups (also no proofs).
3. Eigenvarieties and deformations of Galois representations.
4. Steps 4 and 5.
5. Steps 2 and 3.
6. Step 1.

2 January 14: l-adic representations.

2.1 Definitions

Let K be a perfect field (so all of its extensions are separable, e.g. a finite field or a characteristic 0 field), $G_K = \mathrm{Gal}(\overline{K}/K) = \varprojlim_{K'/K \text{ finite Galois}} \mathrm{Gal}(K'/K)$. G_K is a profinite group. By an l-adic representation of G_K we mean

- an algebraic extension $L/\mathbb{Q}_l$
- a finite dimensional vector space V/L
- a continuous representation $r : G_K \to GL(V)$ (meaning continuous with respect to the l-adic topology on V and the profinite topology on G_K).

Lemma 2.1.1. *Suppose $r : G_K \to GL(V/L)$ is an l-adic representation. Then there is $L \supset L_0 \supset \mathbb{Q}_l$ such that $L_0/\mathbb{Q}_l$ is finite, and an $\mathcal{O}_{L_0}$-lattice $\Lambda_0 \subset V$ (i.e. Λ_0 is a finitely generated $\mathcal{O}_{L_0}$-module such that $\Lambda_0 \otimes_{\mathcal{O}_{L_0}} L \xrightarrow{\sim} V$) such that* $\mathrm{im}(r) \subset GL(\Lambda_0/\mathcal{O}_{L_0})$.

This can be proven using elementary analysis, e.g. the Baire Category Theorem.

Let $\lambda_0 \subset \mathcal{O}_{L_0}$ be a prime, so that $\mathbb{F} = \mathcal{O}_{L_0}/\lambda_0$ is a finite field. Then $\overline{r} : G_K \to GL(\Lambda_0/\lambda_0\Lambda_0)$ may depend on choices (of λ_0), but $\overline{r}^{ss}$ is well-defined.

Example 2.1.2. The cyclotomic character $\epsilon_l : G_K \to \mathbb{Z}_l^\times \subset \overline{\mathbb{Q}}_l^\times$ for $\mathrm{char}(K) \neq l$, defined by

$$\sigma(\zeta) = \zeta^{(\epsilon_l(\sigma) \pmod{l^n})}$$

for any l^nth root of unity ζ and any n.

Example 2.1.3. $H^i_{et}(X_{\overline{K}}, \mathbb{Q}_l)$ if X/K is a variety and $l \neq \mathrm{char}(K)$.

Example 2.1.4. $(T_l E)[1/l]$ if E/K is an elliptic curve with $l \neq \mathrm{char}(K)$, where $T_l E = \varprojlim_n E[l^n](\overline{K})$.

2.2 Weil-Deligne representations

If $K/\mathbb{Q}_p$ is finite with residue field k, and $v_K : K^\times \twoheadrightarrow \mathbb{Z}$ is the valuation normalized so that $|\alpha|_K = (\#k)^{-v_K(\alpha)}$, we have

$$0 \to I_K \to G_K \to G_k \to 0$$

where I_K is the inertia group, and an isomorphism (which we'll also call v_K because it's analogous)

$$\begin{aligned} v_K : G_k &\xrightarrow{\sim} \widehat{\mathbb{Z}} = \varprojlim_N \mathbb{Z}/N \cong \prod_l \mathbb{Z}_L \\ \mathrm{Frob}_K &\mapsto 1 \end{aligned}$$

where Frob_K is the geometric Frobenius, satisfying $(\mathrm{Frob}_K \alpha)^{\#k} = \alpha$ for any $\alpha \in \overline{K}$ (the inverse of the arithmetic Frobenius). Within the above SES we have

$$0 \to I_K \xrightarrow{open} W_K \to \mathrm{Frob}_K^{\mathbb{Z}} \to 0$$

where the Weil group $W_K \hookrightarrow G_K$ is continuous with dense image (but not a homeomorphism onto its image) and we have $v_K : \mathrm{Frob}_K^{\mathbb{Z}} \xrightarrow{\sim} \mathbb{Z}$. I_K has a unique Sylow pro-p subgroup P_K which is normal in I_K, called the wild inertia group. We have an isomorphism

$$t : I_K/P_K \xrightarrow{\sim} \prod_{l \neq p} \mathbb{Z}_l$$

such that $t(\sigma\tau\sigma^{-1}) = (\#k)^{-v_K(\sigma)} t(\tau)$ for $\tau \in I_K$, $\sigma \in W_K$. This induces a map $t_l : I_K/P_K \twoheadrightarrow \mathbb{Z}_l$ which is defined up to $\mathbb{Z}_l^\times$-multiples. Then we have a map

$$\mathrm{art}_K : K^\times \xrightarrow{\sim} W_K^{ab}$$

characterized by the following properties:

1. $v_K \circ \mathrm{art}_K = v_K$ (uniformizers go to lifts of Frob_K)
2. if $\sigma \in \mathrm{Gal}(\overline{K}/\mathbb{Q}_p)$, $\mathrm{art}_{\sigma K} \circ \sigma = \mathrm{conj}_\sigma \circ \mathrm{art}_K$ in $W_{\sigma K}$
3. If K_1/K_2 is finite, we have commutative diagrams

$$\begin{tikzcd} K_1^\times \arrow[r, "\mathrm{art}_{K_1}"] \arrow[d, "N_{K_1/K_2}"'] & W_{K_1}^{ab} \arrow[d] \\ K_2^\times \arrow[r, "\mathrm{art}_{K_2}"] & W_{K_2}^{ab} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} K_1^\times \arrow[r, "\mathrm{art}_{K_1}"] & W_{K_1}^{ab} \\ K_2^\times \arrow[r, "\mathrm{art}_{K_2}"] \arrow[u] & W_{K_2}^{ab} \arrow[u, "\mathrm{tr}"'] \end{tikzcd}$$

   where if $G \supset H$ with H finite index in G then $\mathrm{tr} : G^{ab} \to H^{ab}$ is defined by writing $G = \coprod Hg_i$, and if $g_i g \in Hg_{j(i)}$ for each i, then $\mathrm{tr}(g) := \prod_i (g_i g g_{j(i)}^{-1})$.

This is called the Artin map and the above statements about it are referred to as local class field theory.

Let L be any field of characteristic 0. By a Weil-Deligne (WD) representation of W_K over L, we mean

- a finite dimensional vector space W/L
- a representation $\rho : W_K \to GL(W)$ with open kernel (so there's no continuity assumption—no dependence on the topology on K)
- an element $N \in \mathrm{End}(W)$ such that $\rho(\sigma)N\rho(\sigma)^{-1} = (\#k)^{-v_K(\sigma)}N$ for all $\sigma \in W_K$ (this implies that N is nilpotent).

These things can be put into $\oplus$, Ind, $\otimes$, etc.; note that the way to take tensor products is

$$(W_1, \rho_1, N_1) \otimes (W_2, \rho_2, N_2) = (W_1 \oplus W_2, \rho_1 \otimes \rho_2, N_1 \otimes 1 + 1 \otimes N_2).$$

We say that (W, ρ, N) is

1. Frobenius-semisimple (F-ss) if ρ is semisimple (meaning that $\rho(\sigma)$ semisimple for all $\sigma \in W_K$; equivalently, $\rho(\sigma)$ is semisimple for one $\sigma \in W_K \setminus I_K$);
2. semisimple if it is Frobenius-semisimple and $N = 0$;
3. unitary (sometimes "bounded") if $L/\mathbb{Q}_l$ is algebraic and for all $\sigma \in W_K$, all eigenvalues of $\rho(\sigma)$ are l-adic units; equivalently, there exists $\sigma \in W_K \setminus I_K$ such that all eigenvalues of $\rho(\sigma)$ are l-adic units; equivalently, there is an $\mathcal{O}_L$-lattice $\Lambda \subset W$ preserved by ρ and N.

Proposition 2.2.1. *Suppose $L/\mathbb{Q}_l$ is algebraic, $l \neq p$, $\varphi \in W_K$ is a lift of* Frob_K*, and $t_l : I_K/P_K \twoheadrightarrow \mathbb{Z}_l$ is chosen. Then there is a tensor equivalence of categories $WD = WD_{\varphi,t_l}$ from*

$\{l$-adic representations of G_K over $L\}$ to

$\{$unitary WD representations of W_K over $L\}$, characterized by

$$WD(V, r) = (V, \rho, N)$$

where

$$r(\varphi^m\sigma) = \rho(\varphi^m\sigma)\exp(t_l(\sigma)N)$$

for all $\sigma \in I_K$ and $m \in \mathbb{Z}$.

Given another choice φ', t_l', there is a natural isomorphism $WD_{\varphi,t_l} \xrightarrow{\sim} WD_{\varphi',t_l'}$.

This proposition gives a way of understanding l-adic continuous representations that doesn't involve topology, which means you can compare across different ls.

Example 2.2.2. $WD(\epsilon_l) = (\sigma \mapsto (\#k)^{-v_K(\sigma)}, N = 0) = (|\mathrm{art}_K^{-1}|_K, 0)$.

If (W, ρ, N) is a WD-rep, there is a unique $u \in \mathrm{Aut}(W)$ such that

1. u is unipotent
2. u commutes with $\mathrm{im}(\rho)$ and N

3. $(W, \rho u^{-v_K}, N)$ is Frobenius-semisimple.

$(W, \rho u^{-v_K}, N)$ is called the Frobenius semisimplification of (W, ρ, N), and we write it as $(W, \rho, N)^{F-ss}$. We also write $(W, \rho, N)^{ss} = (W, \rho u^{-v_K}, 0)$.

Conjecture 2.2.3. *Suppose X/K is a smooth proper variety and $l \neq p$. Then*

1. *$WD(H^i_{et}(X_{\overline{K}}, \mathbb{Q}_l))$ is F-ss (known to be true when $i = 1$ or X/K is an abelian variety).*
2. *There exists a WD rep $WD^i(X)$ over $\overline{\mathbb{Q}}$ such that for all $l \neq p$ and $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_l$, we have*
$$WD^i(X) \otimes_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}_l \cong WD(H^i_{et}(X_{\overline{K}}, \mathbb{Q}_l)^{F-ss}$$
(known to be true if X/K is an abelian variety or if X/K has potentially good reduction).

Warning: we can't in general define $WD^i(X)$ over $\mathbb{Q}$, but for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we have $\sigma(WD^i(X)) \cong WD^i(X)$. (So e.g. the trace is defined over $\mathbb{Q}$, but it's not true that a representation of a group is defined over the field cut out by its traces.)

Suppose (W, ρ, N) is a Frobenius-semisimple WD representation and $\sqrt{\#k} \in L$. Then
$$(W, \rho, N) \cong \bigoplus_i (W, \rho_i, 0) \otimes \mathrm{Sp}_{m_i}$$
where ρ_i is irreducible and Sp_m is the WD rep which has underlying vector space $L^{\oplus m}$ with basis $e_1, \ldots, e_m$, N given by $Ne_i = e_{i+1}$ for $i < m$ and $Ne_m = 0$, and $\sigma(e_i) = (\#k)^{(m/2-i)v_K(\sigma)} e_i$ for $\sigma \in W_K$. (For $m > 1$, Sp_m is indecomposable but not irreducible.) Furthermore $\{(\rho_i, m_i)\}$ is uniquely determined.

There is a bijection between

{isomorphism classes of Frobenius-semisimple WD representations over L} and

{finite-dimensional representations R of $W_K \times SL_2$ over L such that for some open subgroup $U \subset W_K$, $R|_{U \times SL_2}$ factors through an algebraic representation of SL_2}

taking (W, ρ, N) to (W, R), where
$$\rho(\sigma) = R\left(\sigma, \begin{pmatrix} (\#k)^{-v(\sigma)/2} & 0 \\ 0 & (\#k)^{v(\sigma)/2} \end{pmatrix}\right)$$
and
$$N = (dR)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
(where dR is the derivative of R, a map from $\mathrm{Lie}(SL_2)$ to $\mathrm{End}(W)$). Warning: while this bijection gives a classification of isomorphism classes of objects, it does not come from an equivalence of categories, because WD reps have more morphisms.

We call a representation R of $W_K \times SL_2$ pure of weight w if for all $\sigma \in W_K$, all eigenvalues α of $R(\sigma, 1)$, and all $\tau : \mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$, we have
$$|\tau(\alpha)| = (\#k(\alpha))^{(w/2)v_K(\sigma)};$$
equivalently, there is some $\sigma \in W_K \setminus I_K$ for which this is true. We call a WD rep (W, ρ, N) pure of weight w if $(W, \rho, N)^{F-ss}$ corresponds to an R which is pure of weight w. (This is not equivalent to ρ being naively pure of weight w; we want to think of Sp_m as being pure of weight 0, even though its Frobenius eigenvalues are all sorts of things.)

Conjecture 2.2.4. *If X/K is smooth and proper with $l \neq p$, then $WD(H^i_{et}(X_{\overline{K}}, \mathbb{Q}_l))$ is pure of weight i (this is known by a famous theorem of Deligne to be true if X has potentially good reduction or if X is an abelian variety).*

2.3 $l = p$

So far we have considered G_K acting on V/L where $K/\mathbb{Q}_p$ is finite and $L/\mathbb{Q}_l$ is algebraic, and $l \neq p$. What about $l = p$? Then there is a topological K-algebra B_{dR} with a continuous action of G_K and a G_K-invariant decreasing filtration Fil^i which is exhaustive (the union of the filtered pieces is all of B_{dR}) and separated (their intersection is 0), such that $\mathrm{gr}^i B_{dR} \cong \widehat{\overline{K}}(\epsilon_p^{i-1})$. We have $B_{dR}^{G_K} = K$.

If (V, r) is a representation of G_K over L, $(V \otimes_{\mathbb{Q}_p} B_{dR})^{G_K}$ is finitely generated over $L \otimes_{\mathbb{Q}_p} K = \prod L_i$, where L_i/L is finite. We have

$$\dim_{L_i}((V \otimes_{\mathbb{Q}_p} B_{dR})^{G_K} \otimes_{L \otimes K} L_i) \leq \dim_L V$$

and we call V de Rham if we have equality for all i. Being de Rham is closed under subobjects, quotients, duals, and tensor products, but not generally under extensions. If $\tau : K \hookrightarrow \overline{L}$ then $HT_\tau(V)$ is the multiset of integers which contains i with multiplicity

$$\dim_{\overline{L}} \mathrm{gr}^i(V \otimes_{\mathbb{Q}_p} B_{dR})^{G_K} \otimes_{L \otimes K, 1 \otimes \tau} \overline{L}.$$

These are the τ-Hodge-Tate numbers of V. V is de Rham if and only if $\#HT_\tau(V) = \dim V$ for all τ. (Note that there is another way of defining Hodge-Tate numbers if V is not de Rham which may have more numbers.) If $\sigma \in G_L$ then $HT_{\sigma\tau}(V) = HT_\tau(V)$.

Example 2.3.1. $HT_\tau(\epsilon_p) = \{-1\}$.

Theorem 2.3.2. *If X/K is smooth and proper, then $H^i_{et}(X_{\overline{K}}, \mathbb{Q}_p)$ is de Rham and*

$$HT_\tau(H^i_{et}(X_{\overline{K}}, \mathbb{Q}_p))$$

contains j with multiplicity $\dim_K H^{i-j}(X, \Omega^j_X)$, where Ω^j_X is the sheaf of holomorphic j-differentials on X. (Note that we don't really need to put a τ since this representation is defined over $\mathbb{Q}_p$, but whatever.)

We would like an analogue of the WD rep classification for $l = p$. This is possible for de Rham representations, as follows. Let $K/\mathbb{Q}_p$ be finite. Let $K_0/\mathbb{Q}_p$ be the maximal unramified subextension (so $K_0/\mathbb{Q}_p$ is unramified and K/K_0 is totally ramified). Suppose K'/K is a Galois extension with maximal unramified subextension K'_0.

Definition 2.3.3. A $(\varphi, N, \mathrm{Gal}(K'/K))$-module over L is

- a finitely generated D over $K'_0 \otimes_{\mathbb{Q}_p} L$ with
- $\varphi : D \xrightarrow{\sim} D$ that is $\mathrm{Frob}_p^{-1} \otimes 1$-semilinear (where $\mathrm{Frob}_p = \mathrm{Frob}_{\mathbb{Q}_p}$),
- $N \in \mathrm{End}(D)$ with $\varphi N = pN\varphi$, and

- a semilinear action ρ of $\mathrm{Gal}(K'/K)$ that commutes with N and φ (where by semilinear we mean $\rho(\sigma)((\alpha \otimes \beta)x) = (\sigma\alpha) \otimes \beta(\rho(\sigma)(x))$, so semilinear with respect to K' but linear with respect to L).

By a filtration $\mathrm{Fil}^\bullet$ on (D, φ, N, ρ) we mean a (decreasing, exhaustive and separated) filtration of $D \otimes_{K_0'} K'$ by $K' \otimes_{\mathbb{Q}_p} L$-submodules which are invariant for the diagonal $\mathrm{Gal}(K'/K)$-action, meaning the action $\sigma(x \otimes \alpha) = \rho(\sigma)(x) \otimes (\sigma\alpha)$.

If we have $K''/K'/K$, a (filtered) $(\varphi, N, \mathrm{Gal}(K'/K))$-module gives rise to a (filtered) $(\varphi, N, \mathrm{Gal}(K''/K))$-module via $D \mapsto D \otimes_{K_0'} K_0''$. (Figure out for yourself what the structure is—it's clear.) Any (filtered) $(\rho, N, \mathrm{Gal}(K'/K))$-module arises in this way from a $(\varphi, N, \mathrm{Gal}(K'''/K))$-module for some $K' \supset K''' \supset K$ where K'''/K is finite.

If (D, φ, N, ρ) is a $(\varphi, N, \mathrm{Gal}(K'/K))$-module, then D has an action of W_K/I_K via $\sigma \mapsto \varphi^{v_{\mathbb{Q}_p}(\sigma)}\sigma$ which is $K_0' \otimes L$-*linear* (not semilinear). Knowing this action is equivalent to knowing the $\mathrm{Gal}(K'/K)$-action.

There is another way of thinking about these things which is done less often but which Richard thinks is more natural. If $\tau : K_0' \hookrightarrow L$, we can define

$$WD_\tau(D, \varphi, N, \rho) = D \otimes_{L \otimes K_0', 1 \otimes \tau} L$$

which is an L-vector space with an action of $W_K/I_{K'}$ and N, i.e. a WD rep of W_K. We have

$$\varphi \otimes 1 : WD_\tau(D, \varphi, N, \rho) \xrightarrow{\sim} WD_{\tau \circ \mathrm{Frob}_p}(D, \varphi, N, \rho).$$

What about the filtration? If $\tau : K' \hookrightarrow L$, and $\mu \in W_{\mathbb{Q}_p}$, we have a map

$$(\mathrm{Fil}^j D \otimes_{K_0'} K') \otimes_{L \otimes K', 1 \otimes \tau\mu} L \hookrightarrow D \otimes_{L \otimes K_0', 1 \otimes \tau\, \mathrm{Frob}_p^{v_p(\mu)}} L$$

and we know that

$$WD_\tau(D) \xrightarrow{\sim, \varphi^{v_p(\mu)} \otimes 1} D \otimes_{L \otimes K_0', 1 \otimes \tau\, \mathrm{Frob}_p^{v_p(\mu)}} L$$

so we obtain a filtration

$$(\mathrm{Fil}^j D \otimes_{K_0'} K') \otimes_{L \otimes K', 1 \otimes \tau\mu} L \twoheadrightarrow \mathrm{Fil}^j_{\tau,\mu} WD_\tau(D) \subset WD_\tau(D).$$

By a filtered WD-rep $(W, \rho, N, \mathrm{Fil}^j_\mu)$ of W_K, we mean a WD rep (W, ρ, N) plus a (decreasing, exhaustive, separated) filtration Fil^j_μ of W for each $\mu \in W_{\mathbb{Q}_p}$ (which doesn't have to commute with the ρ, N actions) such that

$$\mathrm{Fil}^j_{\mu \circ \sigma^{-1}} = \rho(\sigma) \mathrm{Fil}^j_\mu$$

for all $\sigma \in W_K$. (So we're specifying a filtration for each coset of W_K in $W_{\mathbb{Q}_p}$, but without a canonical coset representative.) If $\tau : K' \hookrightarrow L$, then

$$WD_\tau(D, \varphi, N, \rho, \mathrm{Fil}^j)$$

is a filtered WD rep of W_K.

Next time, we'll continue with Galois representations. We'll talk about the relationship between filtered WD reps and de Rham reps, and then the global picture.

3 January 19: p-adic Galois representations.

Recall that we have a finite extension $K/\mathbb{Q}_p$ and we are interested in representations $G_K \to GL(V)$ where V is over L where $L/\mathbb{Q}_l$ is algebraic. We were discussing the case $l = p$, and the subset of de Rham representations.

For K'/K algebraic Galois, we defined filtered $(\varphi, N, \mathrm{Gal}(K'/K))$-modules over L. We let $K'_0 \subset K'$ be the maximal unramified subextension of $\mathbb{Q}_p$; then the data is

- a module D over $K'_0 \otimes_{\mathbb{Q}_p} L$,
- a $\mathrm{Frob}_p^{-1} \otimes 1$-semi-linear endomorphism $\varphi : D \xrightarrow{\sim} D$,
- an $N \in \mathrm{End}(D)$ such that $\varphi N = pN\varphi$,
- a semi-linear action of $\mathrm{Gal}(K'/K)$, and
- a decreasing filtration Fil^i on $D_{K'} = D \otimes_{K'_0} K'$ by $L \otimes K'$-submodules which are invariant under $\mathrm{Gal}(K'/K)$.

We also defined filtered WD reps of W_K over L: the data is

- a WD rep (W, ρ, N) of W_K (where W is a L-vector space, $\rho : W_K \to GL(W)$, and N in $\mathrm{End}(W)$ is such that $\rho(\sigma)N = (\#k)^{-v(\sigma)}N\rho(\sigma)$), and
- Fil^j_μ (for *each* $\mu \in W_{\mathbb{Q}_p}$) is a decreasing, exhaustive, separated filtration such that $\mathrm{Fil}^i_{\mu\sigma} = \rho(\sigma)^{-1}\,\mathrm{Fil}^j_\mu$ for each $\sigma \in W_K$.

If L is sufficiently large, i.e. L contains the image of each embedding $K' \hookrightarrow \overline{L}$ (over $\mathbb{Q}_p$), then there is an equivalence of categories WD_τ (for any given $\tau : K' \hookrightarrow L$) between

{filtered $(\varphi, N, \mathrm{Gal}(K'/K)$-modules over L} and

{filtered WD reps $(\rho, N, \mathrm{Fil}^\bullet_\mu)$ over L such that $\rho|_{I_{K'}} = 1$}.

(If $L \cong \overline{\mathbb{Q}}_l$ then we can take $K' = \overline{K}$.) WD_τ is tensoring by $\otimes_{L\otimes K'_0, 1\otimes\tau} L$; it is independent of τ up to equivalence.

Fontaine preferred the former category because if L is a small field, e.g. $\mathbb{Q}_l$, for example if you're interested in the l-adic cohomology of a variety with coefficients in $\mathbb{Q}_l$ it is more natural. But if you are willing to let the coefficients get big, the objects in the latter category are more concrete.

3.1 Admissibility

Definition 3.1.1. A filtered WD rep $(W, \rho, N, \mathrm{Fil}^\bullet_\mu)$ is admissible if for all sub-WD-reps $W' \subset W$ (invariant by φ and N), we have

$$t_{H,\mu}(w') := e_{K/\mathbb{Q}_p} \sum_j j \dim \mathrm{gr}^j_\mu W' \le t_N(w') := v_p(\det \rho(\varphi)|_{W'})$$

where $v_p(p) = 1$ and $\varphi \mapsto \mathrm{Frob}_K$, with equality if $W' = W$.

Similarly, a filtered $(\varphi, N, \mathrm{Gal}(K'/K))$-module $(D, \varphi, N, \rho, \mathrm{Fil}^\bullet)$ is admissible if for all (φ, N, ρ)-invariant submodules $D' \subset D$,

$$t_H(D') := \sum_j \frac{j \dim_L \mathrm{gr}^j D'_{K'}}{[K' : \mathbb{Q}_p]} \leq \frac{v_p(\det_L(\varphi|_{D'}))}{[K'_0 : \mathbb{Q}_p]} =: t_N(D')$$

with equality if $D' = D$. The two definitions of admissibility coincide under WD_τ.

So we're comparing the valuations of the eigenvalues of Frobenius with where the eigenvectors are situated with respect to the given filtration. The definition may look complicated but it's very concrete in any given case. It provides us a linear algebra interpretation of de Rham representations, as follows.

Theorem 3.1.2. *There is an equivalence of categories $V \mapsto D(V)$, $V(D) \leftarrow D$ between*
$\{$de Rham representations of $G_K/L\}$ and
$\{$admissible filtered (φ, N, G_K)-modules $D/L\}$.
If $L = \overline{\mathbb{Q}}_l$, these are also equivalent to
$\{$admissible filtered WD-reps of W_K over $L\}$
(for smaller L you can get a similar correspondence but you need to impose more conditions, like that some inertia group acts trivially, and/or that the object becomes semistable over a finite extension).

We will call the map from the first to the third thing WD_τ.

So, like when $l \neq p$, we are trying to describe representations in a way that doesn't involve topology, but now it is harder—we can only do de Rham representations and we have to keep track of a filtration. (Note that this theorem includes the assertion that de Rham implies potentially semi-stable, which is hard.)

If $\sigma \in W_{\mathbb{Q}_p}$ and $\tau : \overline{K} \xrightarrow{\sim} L$, the weights $HT_{\tau\sigma}(V)$ contain i with multiplicities

$$\dim_L \mathrm{gr}^i_\sigma WD_\tau(V).$$

(These are captured just by the WD reps, but the positions of the filtrations are continuous invariants which are not.)

Conjecture 3.1.3. *Let X/K be smooth and proper.*

1. *$WD(H^i_{et}(X_{\overline{K}}, \mathbb{Q}_p))$ is F-ss (known if X is an abelian variety and for H^1 in general).*
2. *$WD^i(X) \otimes_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}_p = WD(H^i(X_{\overline{K}}, \overline{\mathbb{Q}}_p))^{F-ss}$ for any $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ (known if X is an abelian variety or if X has good reduction).*

Proposition 3.1.4. *$\chi : G_K \to L^\times$ is de Rham if and only if there is an open subgroup $U \subset K^\times$ and $m_\tau \in \mathbb{Z}$ for all $\tau : K \hookrightarrow \overline{L}$ such that*

$$\chi \circ \mathrm{art}_K |_U = \prod_{\tau : K \hookrightarrow \overline{L}} \tau^{-m_\tau}.$$

In this case, we have $HT_\tau(\chi) = \{m_\tau\}$ and

$$WD(\chi) \circ \mathrm{art}_K = (\chi \circ \mathrm{art}_K) \prod_{\tau : K \hookrightarrow \overline{L}} \tau^{m_\tau}.$$

Example 3.1.5. $WD(\epsilon_p) \circ \mathrm{art}_K = |\cdot|_K$.

Time for global fields!

3.2 Global case

Now let $F/\mathbb{Q}$ be a finite extension.

Conjecture 3.2.1 (Fontaine-Mazur)**.** *An irreducible l-adic representation*

$$r : G_F \to GL_n(\overline{\mathbb{Q}}_l)$$

is a subquotient of $H^i_{et}(X_{\overline{F}}, \overline{\mathbb{Q}}_l) \otimes \epsilon_l^j$ for some X/F smooth projective and some $i \in \mathbb{Z}_{\geq 0}$, $j \in \mathbb{Z}$ if and only if

1. *r is unramified almost everywhere (a.e.), meaning at all but finitely many primes, and*
2. *$r|_{G_{F_v}}$ for $v|l$ is de Rham. (This is why we care about the de Rham condition—it is supposed to characterize Galois representations that arise from geometry.)*

(We will call r satisfying the above two conditions "algebraic".)

We have the map

$$\mathrm{art}_F : \mathbb{A}_F^\times \to G_F^{ab}$$

(where $\mathbb{A}_F^\times = \prod'_v F_v^\times$) which is characterized as follows: it restricts to

$$\mathrm{art}_{F_v} : F_v^\times \to G_{F_v}^{ab}$$

where $F_v^\times \hookrightarrow \mathbb{A}_F^\times$ is the natural embedding and $G_{F_v}^{ab} \hookrightarrow G_F^{ab}$ comes from the decomposition group, and at infinite places $\mathrm{art}_{\mathbb{C}}$ has to be trivial while

$$\mathrm{art}_{\mathbb{R}} : \mathbb{R}^\times \to \mathrm{Gal}(\mathbb{C}/\mathbb{R})$$
$$x \mapsto \begin{cases} 1 & x > 0 \\ c & c < 0. \end{cases}$$

Theorem 3.2.2 (Main theorem of class field theory)**.** $\mathrm{art}_F : \mathbb{A}_F^\times / \overline{F^\times (F_\infty^\times)^\circ} \xrightarrow{\sim} G_F^{ab}$.

Corollary 3.2.3. *$\chi : G_F \to \overline{\mathbb{Q}}_l^\times$ is algebraic if and only if there is an open subgroup $U \subset \mathbb{A}_F^\times$ and $m_\tau \in \mathbb{Z}$ for all $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ such that*

$$\chi \circ \mathrm{art}_F|_U : x \mapsto \prod_\tau (\tau x_l)^{-m_\tau}.$$

(The first condition isn't relevant because such a χ must be unramified almost everywhere. Note that for $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ there is a unique continuous extension of τ to $F_l \to \overline{\mathbb{Q}}_l$, where $F_l = \prod_{v|l} F_v$.)

In this case we have $HT_\tau(\chi) = \{m_\tau\}$.

Corollary 3.2.4. *There is a bijection from*

$\{$algebraic characters $\chi : G_F \to \overline{\mathbb{Q}}_l^\times\}$ to

$\{\chi_0 : \mathbb{A}_F^\times \to \overline{\mathbb{Q}}_l^\times$ with open kernel such that there is $m_\tau \in \mathbb{Z}$ for all $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ with $\chi_0|_{F^\times} = \prod \tau^{m_\tau}\}$.

We have

$$\chi_0(x) = \chi(x) \prod_\tau \tau(x)^{m_\tau}.$$

Such a χ_0 will automatically factor through $\overline{\mathbb{Q}}^\times \subset \overline{\mathbb{Q}}_l^\times$ (because we know that the image of $F^\times$ under χ_0 is algebraic, since χ_0 is just a product of embeddings of $F^\times$ into the field to some power, and $\mathbb{A}_F^\times/F^\times \ker \chi_0$ is finite, so any image of $\mathbb{A}_F^\times$ is a root of an algebraic number). (This is all sort of an attempt to define a global Weil group, but it doesn't work as well outside the abelian case.)

Corollary 3.2.5. *Fix $i : \overline{\mathbb{Q}}_l \cong \mathbb{C}$. Then there is a bijection from*
{algebraic characters $\chi : G_F \to \overline{\mathbb{Q}}_l^\times$} to
{algebraic grossencharacters, i.e. continuous characters

$$\tilde{\chi} : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$$

such that

$$\tilde{\chi}|_{(F_\infty^\times)^\circ} : x \mapsto \prod (\tau x)^{-m_\tau}$$

for some $m_\tau \in \mathbb{Z}$, for all $\tau : F \hookrightarrow \mathbb{C}$}.

This bijection takes an algebraic character χ to χ_0, then to

$$\tilde{\chi}(x) = \chi_0(x) \prod_\tau (\tau x_\infty)^{-m_\tau},$$

where m_τ for $\tilde{\chi}$ corresponds to $HT_{i\circ\tau}(\chi)$. We will denote it $\chi \mapsto \pi_i(\chi)$ and $r_{l,i}(\tilde{\chi}) \leftarrow\!\shortmid \tilde{\chi}$.

This is the model for global Langlands: $\tilde{\chi}$ is an automorphic representation for GL_1 and χ is a 1-dimensional l-adic representation. The proof is direct once you know class field theory, because you have the algebraic χ_0 which is like a representation of the Weil group.

Example 3.2.6. For ϵ_l, we have

$$\epsilon_l \circ \mathrm{art}_F : x \mapsto (N_{F/\mathbb{Q}} x_l) \|x^\infty\| \operatorname{sign}(x_\infty)$$

where for $x \in \mathbb{A}_F^\times$, we have $x_l \in F_l^\times$, $x^\infty \in (\mathbb{A}_F^\infty)^\times$, and $x_\infty \in F_\infty^\times$; and we write $\|x\| = \prod_v |x_v|_v$ (remember all but finitely many terms in the product are 1) and $\operatorname{sign} : \mathbb{R}^\times \twoheadrightarrow \{\pm 1\}$ (if there are several infinite places take the product; at complex places use the identity map). To check this, check it on uniformizers at all places—it's an easy computation.

Then since by definition $N_{F/\mathbb{Q}}(x_l) = \prod_{\tau : F \hookrightarrow \overline{\mathbb{Q}}_l} \tau(x_l)$, and this is the factor we want to get rid of to construct the corresponding algebraic $\epsilon_0 : \mathbb{A}_F^\times \to \mathbb{Q}^\times$, we see that ϵ_0 is $x \mapsto \|x^\infty\| \operatorname{sign}(x_\infty)$ (it's easy to check that this is valued in $\mathbb{Q}^\times$).

Now to find the grossencharacter $\tilde{\epsilon}$, we want it to be trivial on $F^\times$, and we have $\epsilon_0|_{F^\times} = N_{F/\mathbb{Q}}^{-1}$, so removing that gives that $\tilde{\epsilon} : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$ is $x \mapsto \|x\|$. So the absolute value function adele class character correponds to the cyclotomic character.

3.3 CM fields

Definition 3.3.1. We call F CM if there is $c \in \operatorname{Aut}(F)$ such that for all $\tau : F \hookrightarrow \mathbb{C}$, $\tau \circ c = c \circ \tau$. ($c$ is unique if it exists.)

Let $F^+ = F^{c=1}$. We have $[F : F^+] = 1$ or 2. If $[F : F^+] = 1$, we call F totally real; then $c = 1$ and the definition is equivalent to saying that $\tau F \subset \mathbb{R}$ for all $\tau : F \hookrightarrow \mathbb{C}$.

CM fields are closed under subfields and compositums. So if $F/\mathbb{Q}$ is any finite extension, there is $F_0 \subset F$ such that F_0 is CM and if $F_1 \subset F$ is any other CM subfield, then $F_1 \subset F_0$. We call F_0 the maximal CM subfield.

Fact: if $\chi : G_F \to \overline{\mathbb{Q}}_l^\times$ is algebraic and $HT_\tau(\chi) = \{m_\tau\}$, and if $F_0 \subset F$ is the maximal CM subfield, then

1. m_τ only depends on $\tau|_{F_0}$
2. there is $w \in \mathbb{Z}$ (the weight) such that
   (a) if $\tau_1|_{F_0} = \tau_2|_{F_0} \circ c$, then $m_{\tau_1} + m_{\tau_2} = w$, and
   (b) for all v, $WD(\chi|_{G_{F_v}})$ is pure of weight w.

This follows from Dirichlet's unit theorem.

3.4 Starting automorphic forms

First the local theory. Let $K/\mathbb{Q}_p$ be finite and G/K be a (connected) reductive group (e.g. GL_n or a symplectic group; Richard will always mean connected when he says reductive).

Definition 3.4.1. By a smooth representation π of $G(K)$ over $\mathbb{C}$, we mean a $\mathbb{C}$-vector space V (often ∞-dimensional) and a representation $\pi : G(K) \to GL(V)$ such that for all $x \in V$, $\mathrm{Stab}_{G(K)}(x)$ is open in $G(K)$.

We call π admissible if for all open subgroups $U \subset G(K)$, V^U is finite-dimensional.

Theorem 3.4.2 (Bernstein)**.** *Smooth and irreducible implies admissible.*

We have Schur's lemma: if π is irreducible there is a $\chi_\pi : Z(G)(K) \to \mathbb{C}^\times$ with open kernel (where $Z(G)$ is the center of G) such that $\pi|_{Z(G)(K)} = \chi_\pi$.

Definition 3.4.3. The smooth dual (or "contragredient") of π has underlying vector space

$$\pi^\vee = \{f : V \to \mathbb{C} \text{ linear} : \mathrm{Stab}_{G(K)}(f) \text{ is open}\}$$

with the action of $G(K)$ given by $\pi^\vee(g)(f) = f \circ \pi(g^{-1})$. If π is admissible the natural map $\pi \to \pi^{\vee\vee}$ is an isomorphism, and $\pi \mapsto \pi^\vee$ is exact. (In general things are less well-behaved, just as duality for infinite-dimensional vector spaces is less well-behaved than for finite-dimensional ones.)

If $f \in V^\vee$ and $x \in V$, we have a map $G(K) \to \mathbb{C}$ given by $g \mapsto f(\pi(g)x)$. This is called a matrix coefficient (since in the finite-dimensional case it would appear as an entry in a matrix representation of π in a given basis).

If π is irreducible, so that it has a central character χ_π, there is $\chi : G(K) \to \mathbb{R}^\times_{>0}$ smooth such that

$$\chi|_{Z(G)(K)} = |\chi_\pi|.$$

We normalize our matrix coefficients to

$$c_{f,x}(g) = \frac{|f(\pi(g)x)|}{\chi(g)} : G(K)/Z(G)(K) \to \mathbb{R}^\times_{\geq 0}.$$

We call π

- cuspidal (sometimes "supercuspidal") if $c_{f,x}(g)$ has compact support for all f, x;
- discrete series (sometimes "square integrable") if $c_{f,x}$ is square integrable;
- tempered if
$$\int_{G(K)/Z(G)(K)} c_{f,x}(g)^{2+\epsilon} d\mu < \infty$$
for all $\epsilon > 0$, $f \in V^\vee$, and $x \in V$, where μ is the Haar measure on $G(K)$ or $G(K)/Z(G)(K)$, a bi-invariant measure that is unique up to $\mathbb{R}_{>0}^\times$-multiples.

If $P \subset G$ is a parabolic subgroup with unipotent radical N_P, then the Levi subgroup $L_P = P/N_P$ is reductive. For $h \in P(k)$, let
$$\delta_P(h) = |\det(\mathrm{ad}(h)|_{\mathrm{Lie}(N_P)})|_K.$$
If (π, V) is a smooth representation of $L_P(K)$, let
$$n - \mathrm{Ind}_{P(K)}^{G(K)}(\pi, V) = \{\varphi : G(K) \to V \text{ locally constant } | \varphi(hg) = \delta_P(h)^{1/2}\pi(\overline{h})\varphi(g) \\ \text{for all } h \in P(K), g \in G(K)\}$$
where $\overline{h} \in L_P(K)$ is the image of $h \in P(K)$. This is a smooth representation of $G(K)$, and if π is admissible then $n - \mathrm{Ind}_{P(K)}^{G(K)} \pi$ is admissible. Using this induction operator, we can reduce the classification of admissible representations of $G(K)$ to cuspidal representations of $L_P(K)$ for various P.

4 January 21: some local Langlands.

Recall that we had a finite field $K/\mathbb{Q}_p$ and a reductive group G/K, and we were looking at (smooth, admissible) representations (π, V) of $G(K)$. We defined (normalized) matrix coefficients, and cuspidal/discrete series/tempered representations. Given a parabolic subgroup $P \subset G$, we defined $n - \mathrm{Ind}_{P(K)}^{G(K)} \pi$, where π is a representation of $L_P(K)$, where $L_P = P/N_P$.

Fact: any irreducible smooth representation of $G(K)$ is a subquotient of $n - \mathrm{Ind}_{P(K)}^{G(K)} \pi'$ where π' is a cuspidal irreducible representation of $L_P(K)$.

4.1 Hecke algebras

Let $C_c^\infty(G(K)) = \{\varphi : G(K) \to \mathbb{C} \mid \varphi \text{ locally constant and compactly supported}\}$. (Local constancy corresponds to the usual notion of smoothness when we're going from a nonarchimedean field to an archimedean field.) Let $\mathscr{H}(G(K)) = \{\text{complex measures } \varphi\mu \text{ on } G(K)$ where $\varphi \in C_c^\infty(G(K))$ and μ is a Haar measure on $G(K)\}$. This is the Hecke algebra. (Richard thinks it is more natural to think of it as a space of measures rather than functions.) It has a convolution product
$$\varphi_1\mu * \varphi_2\mu = \left(g \mapsto \int_{G(K)} \varphi_1(h)\varphi_2(h^{-1}g) d\mu_h\right) \mu$$

(where by $d\mu_h$ we mean we are integrating with respect to h). It is an associative algebra without identity. If (π, V) is a smooth representation of $G(K)$, $\mathscr{H}(G(K))$ acts on V via

$$\pi(\varphi\mu)x = \int_{G(K)} \varphi(g)\pi(g)(x)d\mu_g$$

for $\varphi\mu \in \mathscr{H}$ and $x \in V$ (this integral is actually a finite sum). If π is admissible then $\mathrm{im}(\pi(\varphi\mu))$ is finite dimensional, because φ is U-bi-invariant for some open compact subgroup U, we have $\mathrm{im}(\pi(\varphi\mu)) \subset V^U$. So $\mathrm{tr}(\pi(\varphi\mu))$ makes sense by restricting to $\mathrm{im}(\pi(\varphi\mu))$, and we get a map

$$\mathrm{tr}\,\pi : \mathscr{H}(G(K)) \to \mathbb{C}$$

(sometimes called a "generalized function", being a linear form on measures). Facts:

1. If $\pi_1, \ldots, \pi_r$ are irreducible and pairwise non-isomorphic, then $\mathrm{tr}\,\pi_1, \ldots, \mathrm{tr}\,\pi_r$ are linearly independent.

2. We have an open dense subgroup $G(K)^{reg} = \{g \in G(K) \mid Z_G(g) \text{ is a torus}\} \subset G(K)$. There is a $\theta_\pi : G(K)^{reg} \to \mathbb{C}$ that is locally constant and locally L^1 on $G(K)$ (but not locally constant or even defined on $G(K)$—we are just asking that any point of $G(K)$ have a neighborhood [in which the complement of $G(K)^{reg}$ is closed nowhere dense] on which θ_π is integrable [but the integral may blow up as you approach the irregular locus]) such that

$$\mathrm{tr}\,\pi(\varphi\mu) = \int_{G(K)^{reg}} \theta_\pi(g)\varphi(g)d\mu_g$$

   for all $\varphi\mu \in \mathscr{H}$.

Note that if for $U \subset G(K)$ we write $\mathscr{H}(G(K) /\!\!/ U)$ for the elements of $G(K)$ bi-invariant by U, then

$$\mathscr{H} = \bigcup_{U \text{ open compact subgroup}} \mathscr{H}(G(K) /\!\!/ U).$$

Unlike $\mathscr{H}$, each subalgebra $\mathscr{H}(G(K) /\!\!/ U)$ has an identity element, the characteristic function of U multiplied by the Haar measure which gives U volume 1.

Proposition 4.1.1. *1. There is a reductive group scheme $\mathscr{G}/\mathcal{O}_K$ with generic fiber G if and only if G has a Borel subgroup B defined over K (it is "quasi-split") and L_B splits (becomes isomorphic to a product of $\mathbb{G}_m$s) over an unramified extension of K; we will call such G unramified.*

2. In this case $\mathscr{G}(\mathcal{O}_K)$ is a maximal compact subgroup of $G(K)$. It is called a hyperspecial subgroup.

If B is a Borel subgroup well positioned with respect to $\mathscr{G}$ (meaning that the group $\mathscr{G}(\mathcal{O}_K)$ defines a point of the apartment associated to L_B in the building associated to G), we have an isomorphism

$$Sat : \mathscr{H}(G(K) /\!\!/ \mathscr{G}(\mathcal{O}_K)) \xrightarrow{\sim} \mathscr{H}(T(K)/\mathscr{T}(\mathcal{O}_K))^{W_K}$$

where $T = L_B$, $\mathscr{T}$ is a torus over $\mathcal{O}_K$ with generic fiber T, and W_K is the Weyl group, by which we mean $W_K = N_G(T)(K)/T(K)$ (if you took the quotient before taking K-points you would get something bigger which is not what we want). (*Sat* is for Satake.) We also have an isomorphism

$$\mathbb{C}[X_*(T)]^{\mathrm{Gal}(\overline{K}/K)} \xrightarrow{\sim} \mathscr{H}(T(K)/\mathscr{T}(\mathcal{O}_K))$$
$$\lambda \mapsto \mathbb{1}_{\mathscr{T}(\mathcal{O}_K)\lambda(\varpi_K)}\mu$$

for $\lambda \in X_*(T)$, where (ϖ_K) is a maximal ideal in $\mathcal{O}_K$ with uniformizer ϖ_K, and μ is a Haar measure such that $\mu(\mathscr{T}(\mathcal{O}_K)) = 1$. We have a formula

$$Sat(\varphi\mu) = \left(t \mapsto \delta_B(t)^{1/2}\int_{N_B(K)} \varphi(tn)d\mu_{N,n}\right)\mu_T$$

where $\mu(\mathscr{G}(\mathcal{O}_K)) = 1$, $\mu_T(\mathscr{T}(\mathcal{O}_K)) = 1$, and $\mu_N(\mathscr{G}(\mathcal{O}_K) \cap N(K)) = 1$. From *Sat*, we see that $\mathscr{H}(G(K) /\!\!/ \mathscr{G}(\mathcal{O}_K))$ is commutative.

We call an irreducible smooth representation π of $G(K)$ unramified with respect to $\mathscr{G}$ if $\pi^{\mathscr{G}(\mathcal{O}_K)} \neq (0)$. This depends on the choice of $\mathscr{G}$! (For GL_n there is a unique choice of $\mathscr{G}$ up to conjugacy, hence a unique meaning of "unramified", but this is not true in general.) In this case (by commutativity of $\mathscr{H}(G(K) /\!\!/ \mathscr{G}(\mathcal{O}_K))$) $\dim \pi^{\mathscr{G}(\mathcal{O}_K)} = 1$, and we get a character

$$\lambda_\pi : \mathscr{H}(G(K) /\!\!/ \mathscr{G}(\mathcal{O}_K)) \to \mathbb{C}$$

such that T acts on $\pi^{\mathscr{G}(\mathcal{O}_K)}$ by $\lambda_\pi(T)$. Then

$$\lambda_\pi \circ Sat^{-1} \in \mathrm{Hom}(X_*(T), \mathbb{C}^\times)/W_K = (X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}^\times)/W_K = \mathrm{Hom}(T(K)/\mathscr{T}(\mathcal{O}_K), \mathbb{C}^\times)/W_K.$$

Facts:

1. If π, π' are unramified WRT $\mathscr{G}$ and $\lambda_\pi = \lambda_{\pi'}$, then $\pi \cong \pi'$.
2. If $\chi \in \mathrm{Hom}(T(K)/\mathscr{T}(\mathcal{O}_K), \mathbb{C}^\times)$, then $n - \mathrm{Ind}_{B(K)}^{G(K)} \chi$ has a unique unramified (WRT $\mathscr{G}$) irreducible subquotient π_χ^{ur}, and

$$\lambda_{\pi_\chi^{ur}} \circ Sat^{-1} = [\chi]$$

   in $\mathrm{Hom}(T(K)/\mathscr{T}(\mathcal{O}_K), \mathbb{C}^\times)/W_K$. Any unramified representation of $G(K)$ arises in this way (because it is determined by its associated λ and we can get any λ this way), and $\pi_{w\chi}^{ur} = \pi_\chi^{ur}$ for any $w \in W_K$.

4.2 Local Langlands for GL_n

For GL_n, we can classify irreducible smooth representations in terms of Galois theoretic data.

Theorem 4.2.1. *There is a bijection* rec *between*
{*irreducible smooth representations of* $GL_n(K)$} *and*
{*n-dimensional F-ss WD reps of* W_K} *such that*

1. *If $n = 1$, $\mathrm{rec}(\pi) \circ \mathrm{art}_K = \pi$.*

2. *π is unramified (for any $\mathscr{G}$, all hyperspecials being conjugate in GL_n) if and only if* $\mathrm{rec}(\pi)$ *is unramified, and*

$$\mathrm{rec}(\pi)(\mathrm{Frob}_K) = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$$

where $\alpha_i = \lambda_\pi \circ Sat^{-1}(\mathrm{diag}(1, \ldots, 1, \varpi_K, 1, \ldots, 1))$ (where the ϖ_K is in the ith spot). (And $N = 0$.)

3.
   - $\mathrm{rec}(\pi)$ *is irreducible if and only if π is cuspidal*
   - $\mathrm{rec}(\pi)$ *is indecomposable if and only if π is discrete series*
   - *π is tempered if and only if* $\mathrm{rec}(\pi)$ *interpreted as a representation of $W_K \times SL_2$ satisfies the following: there is $w \in \mathbb{Z}$ such that for all $\sigma \in W_K$ and all eigenvalues α of $R(\sigma, 1)$, we have $|\alpha| = (\#k)^{wv_K(\sigma)}$. (Note that this is sort of like the definition of purity, although it is only about a given embedding into the complex numbers rather than all embeddings.)*

4. *If $P \subset GL_n$ is a parabolic subgroup with Levi $GL_{n_1} \times \cdots GL_{n_r}$ and π_i is an irreducible smooth representation of $GL_{n_i}(K)$ for all i, and if π is an irreducible subquotient of*

$$n-\mathrm{Ind}_{P(K)}^{GL_n(K)}(\pi_1 \otimes \cdots \otimes \pi_r),$$

then (forgetting N, which you could also say something about but we won't)

$$\mathrm{rec}(\pi)^{ss} \cong \mathrm{rec}(\pi_1)^{ss} \oplus \cdots \oplus \mathrm{rec}(\pi_r)^{ss}.$$

5. $\mathrm{rec}(\pi)^\vee = \mathrm{rec}(\pi^\vee)$.

6. $\mathrm{rec}(\pi \otimes \chi \circ \det) = \mathrm{rec}(\pi) \otimes \mathrm{rec}(\chi)$.

7. *If $\sigma : K' \xrightarrow{\sim} K$ is an isomorphism over $\mathbb{Q}_p$, then $\mathrm{rec}_{K'}(\pi \circ \sigma) = \mathrm{rec}_K(\pi) \circ conj_{\tilde{\sigma}}$, where $\tilde{\sigma} : \overline{K}' \xrightarrow{\sim} \overline{K}$ extends σ.*

8. *Various other properties we don't need right now.*

Here are some constructions that you can do with rec. If their properties aren't clear to you, you should check them yourself—they generally follow pretty directly from the above properties of rec.

Definition 4.2.2. If π_i is an irreducible smooth representation of $GL_{n_i}(K)$, then $\pi_1 \boxplus \pi_2$ is the irreducible smooth representation of $GL_{n_1+n_2}(K)$ defined by

$$\mathrm{rec}(\pi_1 \boxplus \pi_2) = \mathrm{rec}(\pi_1) \oplus \mathrm{rec}(\pi_2).$$

It is an irreducible subquotient of $n-\mathrm{Ind}_{P(K)}^{G(K)}(\pi_1 \otimes \pi_2)$, where $P \subset GL_n(K)$ (for $n = n_1 + n_2$) has Levi $GL_{n_1} \times GL_{n_2}$.

Definition 4.2.3. Let K'/K be finite and π an irreducible smooth rep of $GL_n(K)$. Then we define $BC_K^{K'}(\pi)$ to be the irreducible smooth representation of $GL_n(K')$ with

$$\mathrm{rec}_{K'}(BC_K^{K'}(\pi)) = \mathrm{rec}_K(\pi)|_{W_{K'}}.$$

Note that if $\sigma \in \mathrm{Gal}(K'/K)$ (assuming K'/K is Galois), we have

$$BC_K^{K'}(\pi) \circ \sigma \cong BC_K^{K'}(\pi).$$

$BC_K^{K'}$ takes tempered representations to tempered representations. Furthermore if K'/K is cyclic, with $\mathrm{Gal}(K'/K) = \langle \sigma \rangle$, then

1. An irreducible representation Π of $GL_n(K')$ is in the image of $BC_K^{K'}$ if and only if $\Pi \circ \sigma \cong \Pi$.

2. If π, π' are discrete series representations of $GL_n(K)$ with $BC_K^{K'}(\pi) \cong BC_K^{K'}(\pi')$, then

$$\pi' \cong \pi \otimes (\chi \circ \mathrm{art}_K \circ \det)$$

where $\chi : \mathrm{Gal}(K'/K) \to \mathbb{C}^\times$.

Definition 4.2.4. Again let K'/K be finite and Π an irreducible smooth representation of $GL_n(K')$. Then we define $AI_K^{K'}\Pi$ to be the irreducible smooth representation of $GL_{n[K':K]}(K)$ satisfying

$$\mathrm{rec}_K(AI_K^{K'}\Pi) = \mathrm{Ind}_{W_{K'}}^{W_K} \mathrm{rec}_{K'}(\Pi).$$

This also takes tempered to tempered, and we have

$$(AI_K^{K'}\Pi) \otimes (\chi \circ \mathrm{art}_K \circ \det) \cong AI_K^{K'}(\Pi)$$

if K'/K is Galois and $\chi : \mathrm{Gal}(K'/K) \to \mathbb{C}^\times$. If furthermore $\mathrm{Gal}(K'/K)$ is cyclic and $\mathrm{Gal}(K'/K)^\vee = \langle \chi \rangle$, then

1. An irreducible smooth representation π of $GL_{n[K':K]}(K)$ is in the image of $AI_K^{K'}$ if and only if

$$\pi \cong \pi \otimes (\chi \circ \mathrm{art}_K \circ \det).$$

2. If Π and Π' are discrete series and $AI_K^{K'}\Pi \cong AI_K^{K'}\Pi'$, then $\Pi' \cong \Pi \circ \sigma$ for some $\sigma \in \mathrm{Gal}(K'/K)$.

Definition 4.2.5. Suppose π is an irreducible cuspidal representation of $GL_n(K)$. Then we define a discrete series representation $\mathrm{Sp}_m(\pi)$ of $GL_{nm}(K)$ by

$$\mathrm{rec}(\mathrm{Sp}_m(\pi)) = \mathrm{rec}(\pi) \otimes \mathrm{Sp}_m.$$

If $n = 1$ and π is unramified, $\mathrm{Sp}_m(\pi)$ is often called a Steinberg representation.

Any irreducible smooth representation π of $GL_n(K)$ can be written in the form

$$\pi = \mathrm{Sp}_{m_1}(\pi_1) \boxplus \cdots \boxplus \mathrm{Sp}_{m_r}(\pi_r)$$

where π_i is an irreducible cuspidal representation of $GL_{n_i}(K)$ and $n = \sum_i m_i n_i$. The multiset $\{(m_i, \pi_i)\}$ is uniquely determined by π.

4.3 Archimedean fields

Let $K = \mathbb{R}$ or $\mathbb{C}$. Let G/K be a reductive group, $\mathfrak{g}_0 = \operatorname{Lie} G(K)$, $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$, $\mathfrak{U} = \mathfrak{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$ (universal for maps from $\mathfrak{g}$ to it which send Lie brackets to commutators), and $\mathfrak{Z} = \mathfrak{Z}(\mathfrak{g})$ the center of $\mathfrak{U}$. Let $U_\infty \subset G(K)$ be the maximal compact subgroup, which is unique up to conjugation.

Definition 4.3.1. By a $(\mathfrak{g}, U_\infty)$-module, we mean a vector space $V/\mathbb{C}$ together with

1. $\pi : \mathfrak{g} \to \operatorname{End}(V)$ which is $\mathbb{C}$-linear and sends Lie brackets to commutators, and

2. $\pi : U_\infty \to GL(V)$ a representation which is locally finite and continuous, meaning that if $x \in V$ then the U_∞-translates of x span a finite-dimensional $\mathbb{C}$-vector space W, and $\pi : U_\infty \to GL(W)$ is continuous (therefore smooth), satisfying

3. (a) if $k \in U_\infty$ and $x \in \mathfrak{g}$ then $\pi(k)\pi(x)\pi(k^{-1}) = \pi(\operatorname{ad}(k)(x))$, and
   (b) if $X \in \operatorname{Lie}(U_\infty)$ then $\pi(X)x = \frac{d}{dt}\pi(e^{tX})(x)|_{t=0}$.

The second condition means that we can decompose $V = \bigoplus_\rho V(\rho)$, where ρ runs over irreducible continuous finite dimensional representations of U_∞ and

$$V(\rho) = \operatorname{im}(\operatorname{Hom}_{U_\infty}(\rho, V) \otimes \rho \to V).$$

The $V(\rho)$ are called isotypic components.

V receives an action of $\mathfrak{U}(\mathfrak{g})$ also denoted π.

Definition 4.3.2. We call V admissible if $\dim V(\rho) < \infty$ for all ρ. We call V $\mathfrak{Z}$-finite if for all $v \in V$, the $\mathfrak{Z}$-translates of V lie in a finite-dimensional space.

Facts: if V is irreducible then it is admissible. Also then there is a unique continuous $\chi_\pi : Z(G)(K) \to \mathbb{C}^\times$ such that

1. $\chi_\pi|_{Z(G)(K)\cap U_\infty} = \pi|_{Z(G)(K)\cap U_\infty}$, and

2. $d\chi_\pi = \pi|_{\operatorname{Lie} Z(G)(K)}$.

This is called the central character of π.

In addition, the center of $\mathfrak{U}(\mathfrak{g})$ acts on π and commutes with the actions of U_∞ and $\mathfrak{g}$, so by Schur's lemma we get a character $\xi_\pi : \mathfrak{Z} \to \mathbb{C}$ such that $\pi(X) = \xi_\pi(X)$ if $x \in \mathfrak{Z}$. We need to recall Harish-Chandra's description of $\mathfrak{Z}$. Let

$$G_{\mathbb{C}} = (\operatorname{res}^K_{\mathbb{R}} G) \times_{\mathbb{R}} \mathbb{C}.$$

Let $B = TN \subset G_{\mathbb{C}}$. Then we have

$$\gamma_{HC} : \mathfrak{Z} \xrightarrow{\sim} \operatorname{Sym}(\operatorname{Lie} T)^{W(\mathbb{C})}$$

such that if V_λ is the irreducible algebraic representation of $G_{\mathbb{C}}$ with highest weight $\lambda \in X^*(T)$ (dominant WRT B), and $\rho = \frac{1}{2}$(the sum of the roots WRT B), so that

$$d(\rho + \lambda) = \frac{1}{2} d(2\rho + 2\lambda) : \operatorname{Lie} T \to \mathbb{C}$$

extends to $\operatorname{Sym}(\operatorname{Lie} T) \to \mathbb{C}$, then

$$\xi_{V_\lambda} = d(\lambda + \rho) \circ \gamma_{HC}.$$

5 January 26: global automorphic representations and forms.

5.1 More archimedean representations

Let $K = \mathbb{R}$ or $\mathbb{C}$, and let G/K be a reductive group. We were looking at $(\mathfrak{g}, U_\infty)$-modules where $\mathfrak{g} = (\operatorname{Lie} G(K)) \otimes_{\mathbb{R}} \mathbb{C}$ and $U_\infty \subset G(K)$ is maximal compact. If such a $(\mathfrak{g}, U_\infty)$-module π is irreducible, we defined the central character $\chi_\pi : K^\times \to \mathbb{C}^\times$ and the infinitesimal character $\xi_\pi : \mathfrak{Z} \to \mathbb{C}$. We also characterized

$$\gamma_{HC} : \mathfrak{Z} \xrightarrow{\sim} \operatorname{Sym}(\operatorname{Lie} T)^{W(\mathbb{C})}$$

by requiring that $\xi_{V_\lambda} = d(\rho + \lambda) \circ \gamma_{HC}$. From this, we see that

$$(X^*(T) \otimes_{\mathbb{Z}} \mathbb{C})/W(\mathbb{C}) = \operatorname{Hom}_{\mathbb{C}}(\operatorname{Lie} T, \mathbb{C})/W(\mathbb{C}) \xrightarrow{\sim} \operatorname{Hom}(\mathfrak{Z}, \mathbb{C}).$$

(The first equality is $\lambda \otimes z \mapsto z d\lambda$.)

For example, $\operatorname{Hom}(\mathfrak{Z}(\mathfrak{gl}_n), \mathbb{C})$ corresponds to multisets of n complex numbers $\{s_1, \dots, s_n\}$ via

$$\operatorname{diag}(T_1, \dots, T_n) \mapsto \sum s_i T_i.$$

Then if π is an irreducible $(\mathfrak{gl}_n, O(n))$-module, ξ_π gives rise to the invariant multiset $HC(\pi) = \{s_1, \dots, s_n\}$ where each $s_i \in \mathbb{C}$. Similarly, if π is an irreducible $(\operatorname{res}^{\mathbb{C}}_{\mathbb{R}} \mathfrak{gl}_n, U(n))$-module, since $\operatorname{res}^{\mathbb{C}}_{\mathbb{R}} \mathfrak{gl}_n$ is just two copies of $\mathfrak{gl}_n$ corresponding to the two linear embeddings of $\mathbb{C}$ into itself, ξ_π gives rise to the pair of invariant multisets $HC_1(\pi) = \{s_1, \dots, s_n\}$, $HC(\pi) = \{t_1, \dots, t_n\}$, where each $s_i, t_i \in \mathbb{C}$.

Theorem 5.1.1. *There is a bijection* $\operatorname{rec}_{\mathbb{R}}$ *taking*
{irreducible $(\mathfrak{gl}_n, O(n))$-modules} to
{n-dimensional continuous semisimple representations of $W_{\mathbb{R}}$},
and similarly a bijection $\operatorname{rec}_{\mathbb{C}}$ *taking*
{irreducible $(\operatorname{res}^{\mathbb{C}}_{\mathbb{R}} \mathfrak{gl}_n, U(n))$-modules} to
{n-dimensional continuous semisimple representations of $W_{\mathbb{C}}$}
where $W_{\mathbb{C}} = \mathbb{C}^\times$ and

$$W_{\mathbb{R}} = \langle \mathbb{C}^\times, j \mid j^2 = -1 \in \mathbb{C}^\times, jzj^{-1} = \overline{z} \text{ for all } z \in \mathbb{C}^\times \rangle.$$

Note that we have an isomorphism

$$\begin{aligned} W_{\mathbb{R}}^{ab} &\xrightarrow{\sim} \mathbb{R}^\times \\ z &\mapsto z\overline{z} \\ j &\mapsto -1. \end{aligned}$$

A representation of $\mathbb{C}^\times$ like $\operatorname{rec}_{\mathbb{R}}(\pi)|_{\mathbb{C}^\times}$ must be of the form

$$\operatorname{rec}_{\mathbb{R}}(\pi)|_{\mathbb{C}^\times} : z \mapsto \operatorname{diag}(z^{s_1}\overline{z}^{t_1}, z^{s_2}\overline{z}^{t_2}, \dots)$$

where $s_i, t_i \in \mathbb{C}$. The expression $z^{s_i}\overline{z}^{t_i}$ doesn't make sense in general, but if $s_i - t_i \in \mathbb{Z}$, we can define

$$z^{s_i}\overline{z}^{t_i} := |z|^{s_i+t_i}\left(\frac{z}{|z|}\right)^{s_i-t_i}.$$

Also the image of $\mathrm{rec}_{\mathbb{R}}(\pi)|_{\mathbb{C}^\times}$ has to be invariant under j, so in fact we must have

$$HC(\pi) = \{s_1, \dots, s_n\} = \{t_1, \dots, t_n\}.$$

Similarly

$$\mathrm{rec}_{\mathbb{C}}(\pi) : z \mapsto \mathrm{diag}(z^{s_1}\overline{z}^{t_1}, \dots, z^{s_n}\overline{z}^{t_n})$$

where $s_i, t_i \in \mathbb{C}$ and $s_i - t_i \in \mathbb{Z}$, with no additional constraints. Now

$$HC_1(\pi) = \{s_1, \dots, s_n\}, \qquad HC_c(\pi) = \{t_1, \dots, t_n\}.$$

As before, we can define $\boxplus$, $BC_{\mathbb{R}}^{\mathbb{C}}$, $AI_{\mathbb{R}}^{\mathbb{C}}$, etc. We have

- π is discrete series if and only if $\mathrm{rec}(\pi)$ is irreducible.
- π is tempered if and only if
  - $K = \mathbb{C}$ and $\Re(s_i + t_i)$ is independent of i, or
  - $K = \mathbb{R}$ and

    $$\mathrm{rec}_{\mathbb{R}}(\pi) = \left(\bigoplus_{j=1}^{m} |z|^{r_j}\,\mathrm{sign}^{\delta_j}\right) \oplus \left(\bigoplus_{j=1}^{m'} (\cdots)\right)$$

    where δ_j is 0 or 1, $(\cdots)$ is

    $$\begin{aligned} z &\mapsto \mathrm{diag}(\{z^{s_j}\overline{z}^{t_j}\}, \{z^{t_j}\overline{z}^{s_j}\}) \\ j &\mapsto \begin{pmatrix} 0 & 1 \\ \{(-1)^{s_j - t_j}\} & 0 \end{pmatrix} \end{aligned}$$

    where $s_j - t_j \in \mathbb{Z}_{\neq 0}$ (if it were 0 we would get a split into smaller blocks), and we have

    $$\Re(r_1) = \cdots = \Re(r_m) = \Re(s_1 + t_1) = \cdots = \Re(s_{m'} + t_{m'}).$$

5.2 Global automorphic representations

Let F be a number field and G/F a reductive group with integral model $\mathscr{G}/\mathcal{O}_F$, where $\mathscr{G}$ is reductive over $\mathcal{O}_F[1/N]$ for some N. Let $U \subset G(\mathbb{A}_F^\infty)$ be an open compact subgroup. This means that there is a finite set S of finite places of F, which contains all $v|N$, such that

$$U = \prod_{\substack{v \notin S \\ v \nmid \infty}} \mathscr{G}(\mathcal{O}_{F,v}) \times U_S$$

where $\mathscr{G}(\mathcal{O}_{F,v})$ is hyperspecial maximal compact and $U_S \subset \prod_{v \in S} G(F_v)$ is an open compact subgroup. Fix a maximal compact subgroup $U_\infty \subset G(F_\infty)$.

Definition 5.2.1. By a smooth representation of $G(\mathbb{A}_F)$ we mean a vector space $V/\mathbb{C}$ and maps

$$\pi : G(\mathbb{A}_F^\infty) \times U_\infty \to GL(V)$$
$$\pi : (\operatorname{Lie} G(F_\infty))_{\mathbb{C}} \to \operatorname{End}(V)$$

such that

- if $x \in V$ then $\operatorname{Stab}_{G(\mathbb{A}_F^\infty)} x$ is open in $G(\mathbb{A}_F^\infty)$.
- $\pi|_{U_\infty}$ is locally finite and continuous.
- if $x \in \operatorname{Lie} U_\infty$ then
$$\pi(X)x = \frac{d}{dt}(\pi(e^{tX})x)|_{t=0},$$
and if $g \in G(\mathbb{A}_F^\infty) \times U_\infty$ and $x \in (\operatorname{Lie} G(F_\infty))_{\mathbb{C}}$ then
$$\pi(g)\pi(x)\pi(g^{-1}) = \pi(\operatorname{ad}(g_\infty)(x)).$$

Definition 5.2.2. We call (V, π) admissible if for all open subgroups $U \subset G(\mathbb{A}_F^\infty)$ and irreducible representations ρ of U_∞, we have $\dim_{\mathbb{C}} V^U(\rho) < \infty$.

(Alternatively, you can let $\tilde{\rho}$ be an irreducible finite dimensional representation of $U \times U_\infty$ and then ask that $V = \bigoplus V(\tilde{\rho})$ where each $V(\tilde{\rho})$ is finite-dimensional.)

Remark 2. Richard is not sure whether irreducible and smooth implies admissible in this setting, though it's not very important.

If (V, π) is irreducible and admissible, we have $\pi = \bigotimes_v' \pi_v$ where π_v is an irreducible smooth representation of $G(F_v)$ with π_v unramified (with respect to $\mathscr{G}(\mathcal{O}_{F,v})$) for all but finitely many v. By $\bigotimes_v'$ we mean the restricted tensor product with respect to $\pi_v^{\mathscr{G}(\mathcal{O}_{F,v})}$, which is 1-dimensional for almost all v. The set $\{\pi_v\}$ is uniquely determined by π. Any $\{\pi_v\}$ satisfying the above conditions can arise.

If $G = GL_n$, we can define $\boxplus$, BC_F^E, AI_F^E, etc. locally using this decomposition. For example

$$\pi \boxplus \pi' = \bigotimes_v{}' (\pi_v \boxplus \pi_v'),$$
$$BC_F^E(\pi) = \bigotimes_w{}' BC_{F_w}^{E_w}(\pi_w),$$
$$AI_F^E(\pi) = \bigotimes_v{}' \left(\boxplus_{w|v} AI_{F_v}^{E_w}(\pi_w)\right).$$

If $\tau : F \hookrightarrow \mathbb{C}$, let $v(\tau)|\infty$ be the associated place, so that τ factors as $F \hookrightarrow F_{v(\tau)} \xrightarrow{\tau} \mathbb{C}$ (we will call the second map τ also). Then we may define $HC_\tau(\pi) = HC_\tau(\pi_{v(\tau)}) \subset \mathbb{C}$ for an admissible irreducible representation π of $GL_n(\mathbb{A}_F)$. We call π algebraic if $HC_\tau(\pi) \subset \mathbb{Z}$ for all τ, and regular if $HC_\tau(\pi)$ has n distinct elements for all τ.

5.3 Automorphic forms

Definition 5.3.1. The space of automorphic forms for a reductive group G over a number field F is $\mathscr{A}(G(F)\backslash G(\mathbb{A}_F))$, the space of functions $f : G(F)\backslash G(\mathbb{A}_F) \to \mathbb{C}$ such that

1. there is an open subgroup $U \subset G(\mathbb{A}_F^\infty)$ such that $f(gu) = f(g)$ for all $u \in U$ and $g \in G(\mathbb{A}_F)$.

2. for all $g^\infty \in G(\mathbb{A}_F^\infty)$, the map $G(F_\infty) \to \mathbb{C}$ taking $g_\infty \mapsto f(g^\infty g_\infty)$ is smooth.

3. f is U_∞- and $\mathfrak{Z}$- finite. The first statement means that

$$\{g \mapsto f(gu_\infty) \mid u_\infty \in U_\infty\}$$

spans a finite dimensional space. The second means that since for $X \in \operatorname{Lie} G(F_\infty)$ we have

$$(Xf)(g) = \frac{d}{dt} f(ge^{tX})|_{t=0},$$

we get a corresponding action of $\mathfrak{U}((\operatorname{Lie} G(F_\infty))_{\mathbb{C}})$ on $\mathscr{A}$, and $\{Xf : X \in \mathfrak{Z}\}$ spans a finite dimensional vector space.

4. f has uniformly moderate growth. This means that if we choose a faithful representation $\rho : G \hookrightarrow GL_n$, and for $g \in G(\mathbb{A}_F)$ define

$$\|g\| = \prod_v \max\{|\rho(g)_{ij}|_v, |\rho(g^{-1})_{ij}|_v\},$$

Then there is m such that for all $x \in \mathfrak{U}((\operatorname{Lie} G(F_\infty))_{\mathbb{C}})$, there is C_X such that

$$|(Xf)(g)| \leq C_X \|g\|^m.$$

Note that this definition depends on the choice of U_∞.

$\mathscr{A}(G(F)\backslash G(\mathbb{A}_F))$ is a $G(\mathbb{A}_F)$-module: for $g \in G(\mathbb{A}_F^\infty) \times U_\infty$ we let $(gf)(h) = f(hg)$, and for $X \in \operatorname{Lie} G(F_\infty)_{\mathbb{C}}$ we let $(Xf)(h) = \frac{d}{dt} f(he^{tX})|_{t=0}$. (Note that if $g_\infty \notin U_\infty$, $h \mapsto f(hg)$ is not invariant under U_∞ but a conjugate, so we can't use that definition.)

If $f \in \mathscr{A}(G(F)\backslash G(\mathbb{A}_F))$, there is an ideal J of $\mathfrak{Z}$ such that $Jf = (0)$ and $\dim_{\mathbb{C}} \mathfrak{Z}/J < \infty$. For a given J with $\dim_{\mathbb{C}} \mathfrak{Z}/J < \infty$, $\mathscr{A}(G(F)\backslash G(\mathbb{A}_F))[J]$ is admissible as a $G(\mathbb{A}_F)$-module.

The cusp forms $\mathscr{A}_0(G(F)\backslash G(\mathbb{A}_F)) \subset \mathscr{A}(G(F)\backslash G(\mathbb{A}_F))$ are the ones such that for all parabolic $P \subset G$, we have

$$\int_{N_P(F)\backslash N_P(\mathbb{A}_F)} f(ng)dn = 0.$$

If $\chi : Z(G)(\mathbb{A}_F)/Z(G)(F) \to \mathbb{C}^\times$ is a continuous character, $\mathscr{A}_0(G(F)\backslash G(\mathbb{A}_F), \chi)$ is the subspace where $Z(G)(\mathbb{A}_F)$ acts by χ. We have

$$\mathscr{A}_0(G(F)\backslash G(\mathbb{A}_F)) = \bigoplus_\chi \mathscr{A}_0(G(F)\backslash G(\mathbb{A}_F), \chi) \otimes \operatorname{Sym}\left(X^*(G)_{\mathbb{C}}^{\operatorname{Gal}(\overline{F}/F)}\right)$$

where in the second term we interpret $\chi \in X^*(G)^{\mathrm{Gal}(\overline{F}/F)}$ as multiplication by the element $\log \|\chi(g)\| \in \mathbb{A}_F^\times$ (where for $x \in \mathbb{A}_F^\times$ we define $\|x\| = \prod_v |x_v|_v$). Then we have

$$\mathscr{A}_0(G(F)\backslash G(\mathbb{A}_F), \chi) = \bigoplus_i \pi_i$$

where each π_i is irreducible. By a cuspidal automorphic representation of G we mean an irreducible constituent of some $\mathscr{A}_0(G(F)\backslash G(\mathbb{A}_F), \chi)$, i.e. an irreducible subquotient of $\mathscr{A}_0(G(F)\backslash G(\mathbb{A}_F))$ (since the $\otimes \operatorname{Sym}\left(X^*(G)_{\mathbb{C}}^{\mathrm{Gal}(\overline{F}/F)}\right)$ doesn't introduce new irreducible subquotients, just extensions of existing ones by themselves).

We will be interested in algebraic cuspidal automorphic representations, and in fact will concentrate on polarized regular algebraic cuspidal (PRAC) automorphic representations of $GL_n(\mathbb{A}_F)$, where by polarized we mean that F is a CM field and $\pi \circ c \cong \pi^\vee \| \det \|^{1-n}$. (This is because these are the representations we're good at handling, not because more general ones wouldn't be interesting.)

Theorem 5.3.2. *Suppose F is a CM field and π a PRAC automorphic representation of $GL_n(\mathbb{A}_F)$. Let $i : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. Then there is a unique algebraic semisimple l-adic representation*

$$r(\pi) = r_{l,i}(\pi) : G_F \to GL_n(\overline{\mathbb{Q}}_l)$$

such that

1. *for all $v \nmid \infty$, $i(WD(r(\pi|_{G_{F_v}}))) = \operatorname{rec}(\pi_v)$.*
2. *for each $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$, $HT_\tau(r(\pi)) = -HC_{i\circ\tau}(\pi_\infty)$.*
3. *$r(\pi)^c = r(\pi) \circ \operatorname{conj}_{\tilde{c}}$ (where $\tilde{c} \in \mathrm{Gal}(\overline{F}/F^+) \setminus \mathrm{Gal}(\overline{F}/F)$) satisfies*

   $$r(\pi)^c \cong r(\pi)^\vee \epsilon_l^{1-n}.$$

   In fact, given a place $v|\infty$ of $\overline{F}$ and the corresponding $c_v \in \mathrm{Gal}(\overline{F}/F^+)$, there is a nondegenerate symmetric bilinear pairing

   $$\langle \cdot, \cdot \rangle_v : \overline{\mathbb{Q}}_l^n \times \overline{\mathbb{Q}}_l^n \to \overline{\mathbb{Q}}_l$$

   such that

   $$\langle r(\pi)(\sigma)x, r(\pi)(c_v \sigma c_v^{-1})y \rangle_v = \epsilon_l^{1-n}(\sigma)\langle x, y\rangle_v$$

   for all $x, y \in \overline{\mathbb{Q}}_l^n$ and $\sigma \in G_F$.

The following are equivalent conditions to Item 3 above.

- if $F = F^+$ and n is even, $r(\pi)$ factors through $GSp_n(\overline{\mathbb{Q}}_l) \subset GL_n(\overline{\mathbb{Q}}_l)$ and has multiplier character ϵ_l^{1-n}. This factorization is given by $(x, y) = \langle x, r(\pi)(c_v)y\rangle$ (so that we get an alternating pairing from the given symmetric pairing).
- if $F = F^+$ and n is odd, $r(\pi)$ factors through $GO_n(\overline{\mathbb{Q}}_l) \subset GL_n(\overline{\mathbb{Q}}_l)$ and has multiplier character ϵ_l^{1-n}.

- if $F \neq F^+$, define

$$\mathscr{G}_n = (GL_n \times GL_1) \rtimes \{1, j\}$$

where $j^2 = 1$ and $j(g, \mu)j^{-1} = (\mu(g^{-1})^T, \mu)$. Let $\nu : \mathscr{G}_n \to \mathbb{G}_m$ be the map taking $(g, \mu) \mapsto \mu$ and $j \mapsto -1$. Then $r(\pi)$ extends to a map $\widetilde{r(\pi)} : G_{F^+} \to \mathscr{G}_n(\overline{\mathbb{Q}}_l)$ such that if $\sigma \in G_F$ then $\sigma \mapsto (r(\pi)(\sigma), \epsilon_l(\sigma)^{1-n})$, $\widetilde{r(\pi)}^{-1}\mathscr{G}_n^\circ(\overline{\mathbb{Q}}_l) = G_F$ (so the rest of G_{F^+} goes to j times something), and $\nu \circ \widetilde{r(\pi)} = \epsilon_l^{1-n}\delta_{F/F^+}^n$ where δ_{F/F^+} is the natural character $\mathrm{Gal}(F/F^+) \xrightarrow{\sim} \{\pm 1\}$.

  (Why is this group the way it is? Richard just "used it because it worked", but Buzzard-Gee did write a paper about this. It is sort of like the L-group of the general unitary group, or possibly actually isomorphic to it, just with a different splitting. The complication in general is that there are two different definitions of algebraicity which are off by 1/2 from each other.)

6 January 28: more automorphic reps; intro to (φ, Γ)-modules.

6.1 Finishing automorphic representations on GL_n

If F is a CM field, we defined PRAC (polarized $[\pi^c = \pi^\vee \otimes \|\det\|^{1-n}]$ regular automorphic cuspidal) representations of $GL_n(\mathbb{A}_F)$, and said that we could associate to such a π a Galois representation $r_{l,i}(\pi) = r(\pi) : G_F \to GL_n(\overline{\mathbb{Q}}_l)$ (where $i : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ has been chosen) such that $WD(r(\pi))_{G_{F_v}}^{F-ss} = \mathrm{rec}(\pi_v)$, $HT_\tau(r(\pi)) = -HC_{i\circ\tau}(\pi_\infty)$, and $r^c \cong r^\vee\epsilon_l^{1-n}$. We also said that the last isomorphism came from a symmetric bilinear $\langle\cdot,\cdot\rangle_{c_v} : \overline{\mathbb{Q}}_l^n \times \overline{\mathbb{Q}}_l^n \to \overline{\mathbb{Q}}_l$ such that

$$\langle r(\pi)(\sigma)x, r(\pi)(c_v\sigma c_v^{-1})y\rangle_{c_v} = \epsilon_l^{1-n}(\sigma)\langle x, y\rangle_{c_v},$$

or if you want to describe the isomorphism without picking a particular complex conjugation, if $F = F^+$ you say that $r(\pi)$ factors through GSp_n (for n even) or GO_n (for n odd) with multiplier $\epsilon_l^{1-n}(\sigma)$, and if $F \neq F^+$ you define

$$\mathscr{G}_n = (GL_n \times GL_1) \rtimes \{1, j\} : j(g, \mu)j^{-1} = (\mu(g^{-1})^T, \mu)$$

and say that $r(\pi)$ extends to $\widetilde{r(\pi)} : G_{F^+} \to \mathscr{G}_n(\overline{\mathbb{Q}}_l)$ such that $\widetilde{r(\pi)}(\mathscr{G}_n^0 = GL_n \times GL_1) = G_F$, $\widetilde{r(\pi)}|_{G_F} = (r(\pi), \epsilon_l^{1-n})$, and if $\nu : \mathscr{G}_n \to \mathbb{G}_m$ is defined as last time then $\nu \circ \widetilde{r(\pi)} = \epsilon_l^{1-n}\delta_{F/F^+}^n$. If $r : G_F \to GL_n(\overline{\mathbb{Q}}_l)$ has such a pairing we will call it conjugate self-dual. If r comes from such a π we will call it automorphic.

Theorem 6.1.1. *Let F be a CM field, l a prime, $i : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. Suppose $r : G_F \to GL_n(\overline{\mathbb{Q}}_l)$ is a regular algebraic conjugate self-dual l-adic representation. Suppose also*

- $l > 2(n+1)$
- $\zeta_l \notin F$

- $\overline{r}|_{G_{F(\zeta_l)}}$ *is irreducible*
- $r|_{G_{F_v}}$ *is potentially diagonalizable (a condition stronger than being de Rham, but it's not clear how much stronger; the technical definition is something about being crystalline on a finite extension of* F_v*, and lying on the same component in the universal crystalline deformation space with fixed HT numbers as a diagonal representation. A typical example of a representation that satisfies this is when* l *is non-ramified in* F*,* $r|_{G_{F_v}}$ *is crystalline for all* $v|l$*, and* $HT_\tau(r) \subset [0, l-2]$ *for all* τ*).*

Then there is a finite Galois extension of CM fields F'/F*, linearly disjoint from any given finite extension of* F*, such that* $r|_{G_{F'}}$ *is automorphic (this is called "potentially automorphic").*

Example 6.1.2. Let π be a PRAC automorphic representation of $GL_2(\mathbb{A})$. (These are in bijection with modular newforms of weight $k \geq 2$.) Assume π is not CM (i.e. $\pi \ncong \pi \otimes \chi$ for any continuous character χ of $\mathbb{A}^\times/\mathbb{Q}^\times$; equivalently, π is not an automorphic induction). Then $\operatorname{Sym}^{n-1} r_l(\pi)$ is potentially automorphic for $l \gg 0$. Consequently, $L(\operatorname{Sym}^{n-1}\pi, s)$ has meromorphic continuation to $\mathbb{C}$ and satisfies the expected functional equation.

The point of Newton-Thorne is to remove the "potential" in the theorem for $\operatorname{Sym}^{n-1} r_l(\pi)$, thus replacing "meromorphic" by "holomorphic".

Theorem 6.1.3. *1. If* E/F *is a solvable Galois extension of number fields and* π *is a cuspidal automorphic representation of* $GL_n(\mathbb{A}_F)$*, then there exist a decomposition* $n = n_1 + \cdots + n_r$ *and cuspidal automorphic representations* π_i *of* $GL_{n_i}(\mathbb{A}_E)$ *such that*

$$BC_F^E(\pi) \cong \pi_1 \boxplus \cdots \boxplus \pi_r.$$

2. If E/F *is cyclic of prime degree, say* $\operatorname{Gal}(E/F) = \langle\sigma\rangle$*, and if* Π *is a cuspidal automorphic representation of* $GL_n(\mathbb{A}_E)$*, then* $\Pi \cong BC_F^E(\pi)$ *if and only if* $\Pi\circ\sigma \cong \Pi$*, and in this case* π *is unique up to twisting by a character* $\chi \circ \operatorname{art} \circ \det$ *where* $\chi : \operatorname{Gal}(E/F) \to \mathbb{C}^\times$*.*

Combining this with the theorem at the end of the last lecture and the Chebotarev Density Theorem, we get the following.

Proposition 6.1.4. *Suppose* F *is a CM field and* $r : G_F \to GL_n(\overline{\mathbb{Q}}_l)$ *is a conjugate self-dual, regular algebraic* l*-adic representation. Suppose also that* E/F *is a solvable Galois CM extension such that* $r|_{G_E}$ *is irreducible. Then* r *is automorphic if and only if* $r|_{G_E}$ *is automorphic.*

6.2 Unitary groups

It's hard to do arithmetic on automorphic forms on GL_n, because there's too much real analysis. So we usually work with other groups. In particular, suppose F is an imaginary CM field ($F \neq F^+$), and suppose that F/F^+ is unramified at all finite places (this is just for technical simplicity; it implies that $[F^+ : \mathbb{Q}]$ is even). Let $G_n = G$ be the reductive group scheme over $\mathcal{O}_{F^+}$ defined by

$$G(R) = \{g \in GL_n(\mathcal{O}_F \otimes_{\mathcal{O}_{F^+}} R) \mid g(g^{c\otimes 1})^T = \mathrm{id}_n\}.$$

(This is reductive because F/F^+ being unramified at all finite places means that $\mathcal{O}_F/\mathcal{O}_{F^+}$ is etale.) If $v|\infty$ we have $G_n(F_v) \cong U(n)$, which is compact. If $v = \tilde{v}c(\tilde{v})$ splits in F, we have

$$G(F_v^+) = \{(g_1, g_2) \in GL_n(F_{\tilde{v}}) \times GL_n(F_{c(\tilde{v})}) \mid g_2 = g_1^{-T}\}.$$

This gives us the following diagram, where all the arrows are isomorphisms.

$$\begin{tikzcd}
G(F_v^+) \arrow[r] \arrow[dr] & GL_n(F_{\tilde{v}}) & GL_n(F_v) \arrow[l] \arrow[d, "g \mapsto g^{-T}"'] \\
 & GL_n(F_{c(\tilde{v})}) & GL_n(F_v) \arrow[l]
\end{tikzcd}$$

Theorem 6.2.1 (Labesse)**.** *Choose $\mu : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$ a continuous character such that $\mu^c = \mu^{-1}$ and if $v|\infty$ then $\mu_v(z) = \frac{z}{|z|}$. (This depends on the choice of isomorphism $F_v \xrightarrow{\sim} \mathbb{C}$.) Suppose π is a cuspidal automorphic representation of $G(\mathbb{A}_{F^+})$. Then there is a unique $n = n_1m_1 + \cdots + n_rm_r$ with $n_i, m_i \in \mathbb{Z}_{>0}$ and PRAC automorphic representations π_i of $GL_n(\mathbb{A}_F)$ such that if*

$$\Pi = \boxplus_{i=1}^r \mu^{\delta_i} \|\cdot\|^{(n_i-n)/2} \left(\pi_i \|\cdot\|^{(1-m_i)/2} \boxplus \cdots \boxplus \pi_i \|\cdot\|^{(m_i-1)/2}\right)$$

where δ_i is 0 if $n + n_i + m_i$ is odd and 1 if $n + n_i + m_i$ is even, which is a representation of $GL_n(\mathbb{A}_F)$, then

1. *if $v = \tilde{v}c(\tilde{v})$ in F then*
$$\Pi_{\tilde{v}} \cong \pi_v |\cdot|_v^{(1-n)/2}$$
(using $GL_n(F_{\tilde{v}}) \cong G(F_v^+)$).

2. *if v is inert in F/F^+ and π_v is unramified WRT $G(\mathcal{O}_{F_v^+})$, then Π_v is unramified. We can describe it using the following diagram.*

$$\begin{tikzcd}
(F_v^\times/\mathcal{O}_{F_v}^\times)^{\lfloor n/2 \rfloor} \arrow[r, "\lambda_{\pi_v} \circ Sat^{-1}_{F_v^+, G}"] & \mathbb{C}^\times \\
(F_v^\times/\mathcal{O}_{F_v}^\times)^n \arrow[u, "N : (\alpha_1, \ldots, \alpha_n) \mapsto (\alpha_1/\alpha_n^c, \alpha_2/\alpha_{n-1}^c, \ldots)"] \arrow[ur, "\lambda_{\Pi_v} \circ Sat_{F_v, GL_n}"']
\end{tikzcd}$$

If we have $G \supset B \supset T$ over F_v^+, we can write

$$T \cong \begin{cases} (\mathrm{res}_{F_v^+}^{F_v} \mathbb{G}_m)^{n/2} & n \text{ even} \\ (\mathrm{res}_{F_v^+}^{F_v} \mathbb{G}_m)^{(n-1)/2} \times (\mathrm{res}_{F_v^+}^{F_v} \mathbb{G}_m)^{N=1} & n \text{ odd} \end{cases}$$

and in both cases $T \times_{F_v^+} F_v \cong \mathbb{G}_m^n$.

3. *If $v|\infty$ and the representation π_v of $U(n)$ has highest weight $(a_1, \dots, a_n) \in \mathbb{Z}^n_+$ where $a_1 \geq \dots \geq a_n$, and $\tau : F_v \xrightarrow{\sim} \mathbb{C}$, then*

$$HC_\tau(\Pi_v) = \{a_1, a_2 - 1, \dots, a_n + 1 - n\} \text{ or } \{-a_n, -a_{n-1} - 1, \dots, -a_1 + 1 - n\}.$$

We have

$$T(F_v^+) = \{\operatorname{diag}(z_1, \dots, z_n) \mid z_i c(z_i) = 1 \forall i\}$$

(so $(z_1, \dots, z_n) \in (F_v^\times)^n$), and

$$\operatorname{diag}(z_1, \dots, z_n) \mapsto \prod (\tau z_i)^{a_i}.$$

We call Π the base change of π and write $\Pi = BC^F_{F^+}(\pi)$. Note that Π is independent of the choice of μ, but the π_i do depend on that choice.

Corollary 6.2.2. *Suppose π is a cuspidal automorphic representation of $G(\mathbb{A}_{F^+})$ and $i : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. (Note that self-dual, regular, algebraic, etc. are now automatic because of the compactness and structure of G.) Then there is a semisimple regular algebraic l-adic representation*

$$r(\pi) = r_{l,i}(\pi) : G_F \to GL_n(\overline{\mathbb{Q}}_l)$$

such that

1. *if $v = \tilde{v}c(\tilde{v})$ splits in F then*

$$WD(r(\pi)|_{G_{F_{\tilde{v}}}})^{F-ss} = \operatorname{rec}(\pi_v |\cdot|^{(1-n)/2})$$

where π_v is interpreted as a representation of $GL_n(F_{\tilde{v}}) \cong G(F_v^+)$.

2. *If v is inert in F and π_v is unramified then $WD(r(\pi)|_{G_{F_v}})$ is unramified and takes $\operatorname{Frob}_{F_v}$ to*

$$(\#k(v))^{(n-1)/2} \operatorname{diag}(\lambda_{\pi_v} \circ Sat^{-1}_{F_v}(\varpi_v, 1, \dots, 1), \lambda_{\pi_v} \circ Sat^{-1}_{F_v}(1, \varpi_v, 1, \dots, 1), \dots).$$

3. *$r(\pi)$ is conjugate self-dual.*

4. *if $v|\infty$ and $\tau : F_v \xrightarrow{\sim} \mathbb{C}$ is continuous, then*

$$HT_{i^{-1}\circ\tau}(r(\pi)) = \{-a_1, 1 - a_2, \dots, n - 1 - a_n\}$$

if π_v has highest weight $(a_1, \dots, a_n)$ WRT τ.

Also, if $r(\pi)$ is irreducible, then $BC^F_{F^+}(\pi)$ is cuspidal ($r = 1$, $m_1 = 1$, $n_1 = n$). (This should be iff but we don't know that.)

Theorem 6.2.3. *If Π is a PRAC automorphic representation of $GL_n(\mathbb{A}_F)$, then there is a cuspidal automorphic representation π of $G(\mathbb{A}_{F^+})$ such that $BC^F_{F^+}(\pi) = \Pi$. (This is not usually unique! There's no strong multiplicity one.) Furthermore, we may choose π which is unramified with respect to $G(\mathcal{O}_{F_v^+})$ at any place v where Π_v was unramified.*

6.3 Starting (φ, Γ)-modules

When $l = p$, for $K/\mathbb{Q}_p$, we saw that de Rham p-adic representations of G_K correspond to filtered admissible WD reps. But there are many more p-adic representations of G_K. What do those correspond to? One good answer is (φ, Γ)-modules.

Let $K/\mathbb{Q}_p$ be finite with residue field k and maximal unramified subextension K_0 (so K/K_0 is totally ramified and $K_0/\mathbb{Q}_p$ is unramified).

Let ζ_{p^n} for $n \geq 1$ be a compatible system of p-power roots of 1, so that $\zeta_{p^{n+1}}^p = \zeta_{p^n}$, $\zeta_p \neq 1$, and $\zeta_p^p = 1$. Let $K_n = K(\zeta_{p^n})$ and $K_\infty = \bigcup_n K_n$. We have an embedding ϵ_p : $\mathrm{Gal}(K_\infty/K) \hookrightarrow \mathbb{Z}_p^\times$, whose image is an open subgroup (which is everything if $K = \mathbb{Q}_p$). Let $\Gamma_K = \mathrm{Gal}(K_\infty/K)$ and $H_K = \mathrm{Gal}(\overline{K}/K_\infty)$, so that we have

$$0 \to H_K \to G_K \to \Gamma_K \to 0.$$

We have $K_\infty \supset K_0' \supset \mathbb{Q}_p$ where K_∞/K_0' is totally ramified and $K_0'/\mathbb{Q}_p$ is unramified (and finite). Let k' be the residue field of K_∞ (or K_0'). Let $\tilde{e}_K$ be the ramification index of K_∞ over $\mathbb{Q}_{p,\infty}$, that is,

$$\frac{[K_\infty : \mathbb{Q}_{p,\infty}]}{[K_0' : \mathbb{Q}_p]}$$

which divides but may not equal $e_{K/\mathbb{Q}_p}$.

Let Δ_K be the maximal p-power torsion subgroup of Γ_K; this is $\{1\}$ if $p \neq 2$ and contained in $\{\pm 1\}$ if $p = 2$, so it's pretty silly, but Newton-Thorne crucially uses the case $p = 2$, so we should remember it exists.

In summary, we have towers

$$\overline{K} - K_\infty - K_n - K - K_0 - \mathbb{Q}_p$$

and

$$\overline{K} - K_\infty - K_0' - K_0 - \mathbb{Q}_p$$

with K_n, K, K_0', K_0 being finite over $\mathbb{Q}_p$ and the others being infinite.

7 February 2: setting up (φ, Γ)-modules.

Good references for (φ, Γ)-modules are [6] by Kedlaya, Pottharst, and Xiao, and [1] by Berger.

7.1 Coefficient rings

Our coefficients will be a finite field extension $L/\mathbb{Q}_p$, or more generally an affinoid algebra $A/\mathbb{Q}_p$. (The former is an example of the latter; we will try to mostly work over the latter, but we will say some things that are only true in the former case.) By an affinoid algebra we mean for example a Tate algebra

$$\mathbb{Q}_p\langle T_1, \ldots, T_d\rangle = \left\{ \sum a_{\underline{i}} \underline{T}^{\underline{i}} \mid |a_{\underline{i}}|_p \to 0 \text{ as } \underline{i} \to \infty \right\}$$

or more generally this mod some ideal I, where $a_{\underline{i}} = a_{i_1 i_2 \cdots} T_1^{i_1} T_2^{i_2} \cdots$. Affinoid algebras

- are noetherian
- are Jacobson (every prime ideal is an intersection of maximal ideals)
- have residue fields at maximal ideals that are finite extensions of $\mathbb{Q}_p$
- have all ideals closed
- are naturally Banach algebras
- are such that any $\mathbb{Q}_p$-algebra morphism between affinoid algebras is continuous
- are such that all finitely generated modules over an affinoid algebra have a unique topology making them Banach A-modules.

If $\alpha, \beta \in p^{\mathbb{Q}} \cup \{0\}$ with $\alpha \leq \beta$, and A is an affinoid algebra, we define

$$R_A^{[\alpha,\beta]} = \left\{ \sum_{i=-\infty}^{\infty} a_i T^i \mid a_i \in A, |a_i|\beta^i \to 0 \text{ and } |a_{-i}|\alpha^{-i} \to 0 \text{ as } i \to \infty \right\}.$$

This has a Banach algebra norm

$$|f|_{\alpha,\beta} = \sup_i \max(|a_i|_A \alpha^i, |a_i|_A \beta^i).$$

We can also write

$$R_A^{[\alpha,\beta]} = A\langle T, U, V\rangle / \langle U - p^a T^d, T^e V - p^{-b}\rangle$$

where $\alpha = p^{a/d}$ and $\beta = p^{b/e}$, so it is affinoid, and $\mathrm{Sp}(R_A^{[\alpha,\beta]}) = \Delta[\alpha,\beta]_A$ is an annulus over A of inner radius α and outer radius β. We also define the half-open annulus

$$\Delta[\alpha,\beta)_A = \bigcup_{\alpha \leq \gamma < \beta} \Delta[\alpha,\gamma]_A$$

which is a rigid space but not affinoid (the RHS is an affinoid cover). Its ring of global functions is

$$R_A^{[\alpha,\beta)} = \bigcap_{\alpha \leq \gamma < \beta} R_A^{[\alpha,\gamma]}$$

which is a Fréchet-Stein algebra. This means that for $\gamma_1 < \gamma_2$, $R_A^{[\alpha,\gamma_2]} \to R_A^{[\alpha,\gamma_1]}$ is flat, and we have an isomorphism

$$(R_A^{[\alpha,\gamma_2]})^{\wedge}_{|\cdot|_{\alpha,\gamma_1}} \xrightarrow{\sim} R_A^{[\alpha,\gamma_1]}.$$

We discussed the properties of these algebras last year—see notes at [11]. Write $R_A^{\alpha} = R_A^{[\alpha,1)}$ for short, and let

$$R_A = \varinjlim_{\alpha<1} R_A^{\alpha}.$$

This is called the Robba ring.

Proposition 7.1.1. *Suppose $L/\mathbb{Q}_p$ is finite and let $R = R_L^{[\alpha,\beta]}$, $R_L^{[\alpha,\beta)}$, or R_L. Then*

1. *R is a domain.*

2. *R has the Bézout property: every finitely generated ideal of R_L is principal (but there may exist non-finitely-generated ideals).*

3. *every closed ideal is principal.*

4. *R is adequate: if $a, b \in R$, then we can write $a = a_1 a_2$ where $(a_1, b) = 1$ and for all $a_3 | a_2$ such that a_3 is not a unit, $(a_3, b) \neq 1$.*

5. *a finitely generated torsion-free R-module is free.*

6. *if $M \subset R^{\oplus n}$ is a finitely generated submodule, then there is a basis $e_1, \ldots, e_n$ of $R^{\oplus n}$ and $f_1 | f_2 | \cdots | f_d \in R$ such that M has basis $f_1 e_1, \ldots, f_d e_d$. Furthermore, $(f_1), \ldots, (f_d)$ are unique.*

Proof. 1-4 were proved by Lazard in [7] (1962). The fact that 5 and 6 follow from the Bézout property and adequacy was shown by Helmer in [5] (1943) (who was just going through the structure theorem for finitely generated modules over PIDs and looking for the minimal conditions that would make it work). □

Remark 3. Why isn't R noetherian? Take some $f \not\equiv 0$ on $\Delta[\alpha, \beta)$ with infinitely many zeros $z_1, z_2, \ldots$. Then we have an infinite ascending chain

$$(f(T)) \subsetneq \left(\frac{f(T)}{(T - z_1)}\right) \subsetneq \left(\frac{f(T)}{(T - z_1)(T - z_2)}\right) \subsetneq \cdots .$$

For example, on $\Delta[\alpha, 1)$, we can choose

$$f(T) = \log(1 + T) = \sum_{i=1}^{\infty} (-1)^{i-1} \frac{T^i}{i}$$

which converges on the open disc and vanishes at $\zeta - 1$ for any p-power root ζ of 1.

7.2 Coadmissible modules

Given $M \subset R^{\oplus n}$, let

$$M^{sat} = \{x \in R^m \mid \text{there is } 0 \neq f \in R \text{ with } fx \in M\}.$$

This is called the saturation of M. We call M saturated if $M^{sat} = M$; this is equivalent to M being a direct summand of $R^{\oplus n}$. (Why? Direct summands being saturated is clear. In the other direction, if M is saturated, then

$$0 \to M \to R^{\oplus n} \to R^{\oplus n}/M \to 0$$

splits, because M being saturated means that $R^{\oplus n}/M$ is torsion-free and finitely generated, hence free by Proposition 7.1.1.)

A coherent sheaf $\mathscr{M}$ over $R_A^{[\alpha,\beta)}$ is the data of a finitely generated $R_A^{[\alpha,\gamma]}$-module $\mathscr{M}_\gamma$ for all $\gamma \in p^{\mathbb{Q}} \cap [\alpha, \beta)$ together with compatible isomorphisms for all $\gamma_2 > \gamma_1$

$$\mathscr{M}_{\gamma_2} \otimes_{R_A^{[\alpha,\gamma_2]}} R_A^{[\alpha,\gamma_1]} \xrightarrow{\sim} M_{\gamma_1}.$$

We call $\mathscr{M}$ a vector bundle iff each $\mathscr{M}_\gamma$ is flat, or equivalently locally free, or equivalently projective (since $\mathscr{M}_\gamma$ is finitely generated over the noetherian $R_A^{[\alpha,\gamma]}$). We write

$$\Gamma(\mathscr{M}) = \varprojlim \mathscr{M}_\gamma;$$

this is a module over $R_A^{[\alpha,\beta)}$. We call $R_A^{[\alpha,\beta)}$-modules arising this way coadmissible.

The following facts and properties are proved in [11].

- Coherent sheaves over $R_A^{[\alpha,\beta)}$ form an abelian category.
- Γ is an equivalence of categories between coherent sheaves and coadmissible modules. We have
$$\Gamma(\mathscr{M}) \otimes_{R_A^{[\alpha,\beta)}} R_A^{[\alpha,\gamma]} \xrightarrow{\sim} \mathscr{M}_\gamma.$$
- Finitely presented $R_A^{[\alpha,\beta)}$-modules are coadmissible.
- $\Gamma(\mathscr{M})$ is naturally a Fréchet space.
- Morphisms between coadmissible modules are continuous.
- If $M \supset N$ are both coadmissible then N is closed in M.
- If M is coadmissible and $N \subset M$ is closed, then N and M/N are coadmissible.

By a coadmissible module over R_A we mean a module of the form $M \otimes_{R_A^\alpha} R_A$ for some α and some coadmissible M over R_A^α.

7.3 (φ, Γ)-modules

Let $K/\mathbb{Q}_p$ be finite and $K \supset K_0 \supset \mathbb{Q}_p$ be the maximal unramified subextension; let ζ_{p^n} for $n \geq 1$ be such that $\zeta_{p^{n+1}}^p = \zeta_{p^n}$, $\zeta_p \neq 1$, $\zeta_p^p = 1$, all in $\overline{K}$. Let $K_n = K(\zeta_{p^n})$ and $K_\infty = \bigcup_n K_n$. Let $K_\infty \supset K_{\infty,0} \supset \mathbb{Q}_p$ be the (finite!) maximal unramified subextension (this is a change of notation from last time, sorry). Let $\Gamma_K = \mathrm{Gal}(K_\infty/K) \hookrightarrow^{\epsilon_p} \mathbb{Z}_p^\times$ and $H_K = \mathrm{Gal}(\overline{K}/K_\infty)$. Let e_{K_∞} be the ramification index of $K_\infty/\mathbb{Q}_{p,\infty}$, that is,

$$\frac{[K_\infty : \mathbb{Q}_{p,\infty}]}{[K_{\infty,0} : \mathbb{Q}_p]}.$$

(Similarly, for any K'/K we can define $e_{K'_\infty/K_\infty}$.)

To certain additional data π_K, we associate

$$R_A^{[\alpha,\beta]}(\pi_K) = R_A^{[\alpha,\beta]} \otimes_{\mathbb{Q}_p} K_{\infty,0} = R_{A\otimes K_{\infty,0}}^{[\alpha,\beta]}$$

together with

1. a continuous action of Γ_K which is A-linear and $K_{\infty,0}$-semilinear,
2. a map $\varphi : R_A^{[\alpha,\beta]}(\pi_K) \to R_A^{[\alpha^{1/p},\beta^{1/p}]}(\pi_K)$ which is A-linear and $\mathrm{Frob}_p^{-1} = \mathrm{Frob}_{\mathbb{Q}_p}^{-1}$-linear for $K_{\infty,0}$, making $R_A^{[\alpha^{1/p},\beta^{1/p}]}(\pi_K)$ a finite free $R_A^{[\alpha,\beta]}(\pi_K)$-module of rank p.

Given π_K, π'_K, for α, β sufficiently close to 1, there is a canonical isomorphism $R_A^{[\alpha,\beta]}(\pi_K) \cong R_A^{[\alpha,\beta]}(\pi'_K)$ compatible with the additional structure. (It is possible to give an abstract definition of this data without choosing π_K, but that is hard to work with. If you want to write down precise formulas, you need to choose some π_K.)

We can do the same for $R_A^\alpha(\pi_K)$ and $R_A(\pi_K)$.

If K'/K is finite and α sufficiently close to 1, then there is a canonical map

$$R_A^\alpha(\pi_K) \to R_A^{\alpha^{1/e_{K'_\infty/K_\infty}}}(\pi'_{K'})$$

which is $(\varphi, \Gamma_{K'})$-equivariant and finite étale of rank $[K'_\infty : K_\infty]$. If K'/K is Galois then this map is Galois with group $H_{K'}/H_K$, and the $\Gamma_{K'}$-action on $R_A^{\alpha^{1/e_{K'_\infty/K_\infty}}}(\pi'_{K'})$ extends to an action of $G_K/H_{K'}$.

Example 7.3.1. Let $K = \mathbb{Q}_p$. There is a natural choice $\pi^0_{\mathbb{Q}_p}$ (which depends on the choice of ζ_{p^n}) so that for $? = [\alpha, \beta], \alpha, -$, $R_A^?(\pi^0_{\mathbb{Q}_p})$ has

$$\varphi(T) = (1+T)^p - 1$$
$$\gamma(T) = (1+T)^{\epsilon_p(\gamma)} - 1$$

for $\gamma \in \Gamma_{\mathbb{Q}_p}$. If $t = \log(1+T) \in R_A^?(\pi^0_{\mathbb{Q}_p})$, these become $\varphi t = pt$ and $\gamma t = \epsilon_p(\gamma) t$. Let

$$q(T) = \frac{(1+T)^p - 1}{T}.$$

This is degree $p-1$ and has roots $\zeta - 1$ for ζ a primitive pth root of 1. We have

$$\varphi^{n-1}(q) = \frac{(1+T)^{p^n} - 1}{(1+T)^{p^{n-1}} - 1}$$

and this has roots $\zeta - 1$ for ζ a primitive p^nth root of 1. We can also write

$$t = T \prod_{n \geq 1} \frac{\varphi^{n-1}(q)}{p},$$

since the Nth partial product is

$$\frac{(1+T)^{p^N} - 1}{p^N} = \sum_{i=1}^{p^N} \frac{1}{p^N} \binom{p^N}{i} T^i$$

and

$$\frac{1}{p^N} \binom{p^N}{i} = \frac{1}{p^N} \cdot \frac{p^N(p^N-1)\cdots(p^N-(i-1))}{i \cdot 1 \cdot \cdots \cdot (i-1)} \equiv \frac{(-1)^{i-1}}{i} \pmod{p^{N-v_p(i)}\mathbb{Z}_p}$$

so indeed the partial products approach the expansion of log. From this we see that the roots of t are $\zeta - 1$ where ζ is a p-power root of 1.

In general, for $n \gg_\alpha 0$, we have $R_A^\alpha(\pi_K)/\varphi^{n-1}(q) \cong A \otimes_{\mathbb{Q}_p} K_n$. (This is clear if $K = \mathbb{Q}_p$; we can't explain it yet in general.) Any finitely generated ideal of $R^\alpha_{\mathbb{Q}_p}(\pi^0_{\mathbb{Q}_p})$ dividing t^N is of the form

$$\prod \left(\frac{\varphi^{n-1}(q)}{p}\right)^{j_n}$$

for $j_n \in \mathbb{Z} \cap [0, N]$.

There is an operator ∇ on $R^?_A(\pi_K)$ given by

$$\nabla f = \frac{\log(\gamma)(f)}{\log \epsilon_p(\gamma)}$$

for any $\gamma \in \Gamma_K \setminus \{1\}$ sufficiently close to 1, where by definition $\log \gamma = \sum \frac{(\gamma-1)^i(-1)^{i-1}}{i}$. ∇ is $A \otimes_{K_{\infty,0}}$-linear, commutes with φ and Γ_K, and satisfies the Leibniz rule

$$\nabla(fg) = (\nabla f)g + f(\nabla g).$$

Remark 4. This is analogous to how if a Lie group Γ acts on a vector space V, then $\operatorname{Lie}\Gamma$ also acts on V via

$$Xv = \frac{d}{dt}(\exp(tX)v)|_{t=0}$$

and if we let $\exp(tX) = \gamma_t$, then for t sufficiently close to 0 we have by definition

$$Xv = \frac{\log(\exp tX)v}{t} = \frac{(\log \gamma_t)v}{t}$$

so differentiating in the direction of γ_t is the same as acting by $\log \gamma_t$.

Example 7.3.2. On $R_A^\alpha(\pi^0_{\mathbb{Q}_p})$, $\nabla f = t(1+T)\frac{df}{dT}$.

(Justification: we have $\gamma(t) = \epsilon_p(\gamma)t$, so $\gamma(t^n) = \epsilon_p(\gamma)^n t^n$, so

$$\frac{(\log \gamma)(t^n)}{\log(\epsilon_p(\gamma))} = \frac{(\log(\epsilon_p(\gamma)^n))(t^n)}{\log(\epsilon_p(\gamma))} = nt^n = t\frac{d}{dt}(t^n) = t(1+T)\frac{d}{dT}(t^n)$$

for all n.)

If $n \gg_\alpha 0$, there is an expansion map

$$\begin{aligned} \iota_n : R_L^\alpha(\pi_K) &\hookrightarrow (L \otimes K_n)[\![\tilde{t}]\!] \\ t &\mapsto \tilde{t}/p^n \end{aligned}$$

which is $1 \otimes \operatorname{Frob}_p^n$-semilinear for $L \otimes K_{\infty,0}$.

Example 7.3.3. In the case of $R_L^\alpha(\pi^0_{\mathbb{Q}_p})$, $\iota_n(f)$ is the expansion of f at $\zeta_{p^n} - 1$ in terms of $\tilde{t} = p^n t$ (which has a simple zero at $\zeta_{p^n} - 1$).

The map ι_n has the following properties.

1. If $\alpha < \beta$, ι_n on $R^\alpha_L(\pi_K)$ and $R^\beta_L(\pi_K)$ commutes with $R^\alpha_L(\pi_K) \to R^\beta_L(\pi_K)$ for $n \gg 0$.

2. $$\begin{tikzcd} R_L^{\alpha}(\pi_K) \arrow[r, "\iota_n"] \arrow[d, "\varphi"'] & (L \otimes K_n)[\![\tilde{t}]\!] \arrow[d, "\text{embedding } \tilde{t} \mapsto \tilde{t}"] \\ R_L^{\alpha^{1/p}}(\pi_K) \arrow[r, "\iota_{n+1}"] & (L \otimes K_{n+1})[\![\tilde{t}]\!] \end{tikzcd}$$

   commutes for $n \gg 0$.

3. if K'/K is finite,

   $$\begin{tikzcd} R_L^{\alpha}(\pi_K) \arrow[r, "\iota_n"] \arrow[d] & (L \otimes K_n)[\![\tilde{t}]\!] \arrow[d, "\text{natural inclusion}"] \\ R_L^{\alpha^{1/e_{K'_\infty/K_\infty}}}(\pi_{K'}) \arrow[r, "\iota_n"] & (L \otimes K'_n)[\![\tilde{t}]\!] \end{tikzcd}$$

   commutes for $n \gg 0$.

4. ι_n is equivariant for the Γ_K-action, where

   $$\gamma\tilde{t} = \epsilon_p(\gamma)\tilde{t}$$
   $$\nabla = \tilde{t}\frac{d}{d\tilde{t}}$$

   on $(L \otimes K_n)[\![\tilde{t}]\!]$.

Modules over the Robba ring with (φ, Γ_K)-actions will turn out to classify Galois representations for K. Why? It's not really clear...

8 February 9: basics of (φ, Γ)-modules.

8.1 Recap

Recall that we have a finite extension $K/\mathbb{Q}_p$, and additional extensions $K_n = K(\zeta_{p^n})$, $K_\infty = \bigcup K_n$, $\Gamma_K = \mathrm{Gal}(K_\infty/K) \hookrightarrow \mathbb{Z}_p^\times$, $H_K = \mathrm{Gal}(\overline{K}/K_\infty)$, and $K_{\infty,0}$ and K_0, the maximal unramified subextensions of K_∞ and K. Let A be an affinoid algebra over $\mathbb{Q}_p$, and L a finite field extension of $\mathbb{Q}_p$.

For $\alpha, \beta \in p^{\mathbb{Q}}$ with $\alpha \leq \beta$, we defined $R_A^{[\alpha,\beta]}$, the rigid analytic functions on the closed annulus with inner radius α and outer radius β over A. We also defined $R_A^\alpha = \varprojlim_{\alpha \leq \beta < 1} R_A^{[\alpha,\beta]}$, the functions on the half-open annulus with inner radius α and outer radius 1, open on the outside. Finally, we defined $R_A = \bigcup_{\alpha<1} R_A^\alpha$. We saw that in the case $A = L$, for finitely generated ideals and modules, these rings behave like PIDs.

We also saw that modules over R_A^α correspond to coherent sheaves, and coadmissible modules correspond to compatible collections of modules $\mathscr{M}_\gamma$ over $R_A^{[\alpha,\gamma]}$, each finitely generated over $R_A^{[\alpha,\gamma]}$, for all $\alpha \leq \gamma < 1$. We called a coadmissible module $\mathscr{M}$ a vector bundle if each $\mathscr{M}_\gamma$ is flat.

Given some data π_K, we defined a structure $R_A^{[\alpha,\beta]}(\pi_K) = R_A^{[\alpha,\beta]} \otimes_{\mathbb{Q}_p} K_{\infty,0}$, which came with a semilinear Γ_K-action and a $\varphi : R_A^{[\alpha,\beta]}(\pi_K) \to R_A^{[\alpha^{1/p},\beta^{1/p}]}$ which is Frob_p^{-1}-semilinear. (Note that assuming $\alpha \le \beta < 1$, $\alpha^{1/p}, \beta^{1/p}$ are closer to 1 than α, β are, so we're pushing the annulus out toward 1.) We said that for α, β sufficiently close to 1, this thing was canonically independent of π_K, only depending on K. We can similarly define $R_A^{\alpha}(\pi_K), R_A(\pi_K)$.

We saw that when $K = \mathbb{Q}_p$, and we choose a particular simple $\pi_{\mathbb{Q}_p}^{\bullet}$, we have $R_A^{\bullet}(\pi_{\mathbb{Q}_p}^{\bullet}) = R_A^{\bullet}$, $\varphi(T) = (1+T)^p - 1$, and $\gamma(T) = (1+T)^{\epsilon_p(\gamma)} - 1$. If $t = \log(1+T)$ this becomes $\varphi t = pt$ and $\gamma t = \epsilon_p(\gamma)t$. The zeros of t are the p-power roots of 1. We defined an endomorphism ∇ of $R_A^{\bullet}(\pi_K)$ given by

$$f \mapsto \frac{\log(\gamma) f}{\log \epsilon_p(\gamma)}.$$

Here is another explanation of why ∇ looks like this. Suppose we have a map $\rho : G \to GL_n$ of Lie groups. We can differentiate it to get $d\rho : \operatorname{Lie} G \to \operatorname{Lie} GL_n$. We have maps $\exp : \operatorname{Lie} G \to G$ and $\exp : \operatorname{Lie} GL_n \to GL_n$, which commute, so that

$$\rho(\exp(tX)) = \exp(t d\rho(X)).$$

Taking logs of both sides and dividing by t gives

$$\frac{\log \rho(\exp tX)}{t} = d\rho(X).$$

In the case where G is an open subgroup of $\mathbb{Z}_p^{\times}$, so that $\operatorname{Lie} G \subset \mathbb{Z}_p$, if we let $\gamma = \exp(tX)$, this becomes

$$\log \rho(\gamma) = d\rho(\log \gamma) = (\log \gamma) d\rho(1).$$

So our expression for ∇ looks like $d\rho(1) = \frac{\log \rho(\gamma)}{\log \gamma}$. For $n \gg_{\alpha} 0$, we found maps

$$\iota_n : R_A^{\alpha}(\pi_K) \hookrightarrow (A \otimes K_n)[\![\tilde{t}]\!]$$

compatible under φ with

$$\iota_{n+1} : R_A^{\alpha^{1/p}}(\pi_K) \hookrightarrow (A \otimes K_{n+1})[\![\tilde{t}]\!]$$

taking $t \mapsto \tilde{t}/p^n$, with the result that $\gamma \tilde{t} = \epsilon_p(\gamma)\tilde{t}$, $\nabla = \tilde{t}\frac{d}{d\tilde{t}}$. These maps are obtained by taking power series expansions around a p^nth root of unity very close to the outer boundary of radius 1 (so that it lies in our thin annulus).

8.2 Some properties of $R_A(\pi_K)$

Here is a lemma used in Newton-Thorne for which no reference is given.

Lemma 8.2.1. *Let $L/\mathbb{Q}_p$ be finite, $\gamma \in \Gamma_K \backslash \{1\}$, and $f \in R_L(\pi_K)$. Suppose that $(\gamma - 1)^n f = 0$ for some $n > 0$. Then $f \in L \otimes_{\mathbb{Q}_p} K_{\infty,0}$.*

Proof. WLOG $L = \mathbb{Q}_p$, since $R_L(\pi_K) = R_{\mathbb{Q}_p}(\pi_K) \otimes_{\mathbb{Q}_p} L$ for L finite. We induct on n.

If $n = 1$, γ has an infinite orbit in $\mathrm{Sp}(R_{\mathbb{Q}_p}^{[\alpha,\beta]}(\pi_K)) = \Delta(\alpha, \beta)$. This is because we have a finite étale map

$$\mathrm{Sp}(R_{\mathbb{Q}_p}^{[\alpha,\beta]}(\pi_K)) \to \mathrm{Sp}\, R_{\mathbb{Q}_p}^{[\alpha^{e_{K_\infty}}, \beta^{e_{K_\infty}}]}(\pi_{\mathbb{Q}_p}^{\bullet})$$

(where e_{K_∞} is the ramification index of K_∞ over $\mathbb{Q}_{p^\infty}$) and on the RHS we know that

$$\gamma^n x = (1+x)^{\epsilon(\gamma^n)} - 1.$$

Thus if x is in a finite orbit, we have

$$\begin{aligned} \gamma^n x &= x \\ \iff x + 1 &= (1+x)^{\epsilon(\gamma^n)} \\ \iff 1 &= (1+x)^{\epsilon(\gamma^n)-1} \\ \iff 1 &= (1+x)^{p^{v_p(\epsilon(\gamma^n)-1)}} \end{aligned}$$

which is to say that $x = \zeta - 1$ for a p-power root of unity ζ. This is not the case for all x. So if $\gamma f = f$ then there is c such that $f(x) = c$ for infinitely many $x \in \Delta[\alpha, \beta]$, so $f \equiv c$ (rigid analytic functions can't have infinitely zeros on affinoids).

If $n > 1$, γ has a finite orbit on $\Delta[\alpha, 1) = \mathrm{Sp}(R_{\mathbb{Q}_p}^{\alpha}(\pi_K))$, because we have a finite map $\Delta[\alpha, 1) \to \Delta[\alpha^{e_{K_\infty}}, 1) (= \mathrm{Sp}(R_{\mathbb{Q}_p}^{\alpha^{e_{K_\infty}}}(\pi_{\mathbb{Q}_p}^{\bullet})))$, and if ζ is a p-power root of 1 then $\zeta - 1$ has a finite orbit in the target, which gives rise to a finite orbit in the preimage.

Call this finite orbit $z, \gamma z, \ldots, \gamma^{n-1} z, \gamma^n z = z$. If $(\gamma - 1)^n f = 0$, then by the inductive hypothesis, $(\gamma - 1) f = c$ is a constant. Therefore we find

$$\begin{aligned} f(\gamma z) &= f(z) + c \\ f(\gamma^2 z) &= f(\gamma z) + c = f(z) + 2c \\ &\vdots \\ f(z) = f(\gamma^n z) &= f(z) + nc. \end{aligned}$$

We conclude that $c = 0$, so $\gamma f = f$, so by the $n = 1$ case, $f = c$. $\square$

We also need to introduce another extension of $R_A(\pi_K)$. Let $R_A(\pi_K)[\ell]$ be a polynomial ring over $R_A(\pi_K)$, equipped with

$$\varphi(\ell) = p\ell + \log\left(\frac{q}{T}\right), \qquad \gamma(\ell) = \ell + \log\left(\frac{\gamma(T)}{T}\right),$$

where $T \in R_A(\pi_{\mathbb{Q}_p}^{\bullet}) \to R_A(\pi_K)$ and $q = \frac{(1+T)^p - 1}{T}$. We have another operator N on $R_A(\pi_K)[\ell]$ which is $R_A(\pi_K)$-linear and satisfies $N(\ell) = -\frac{p}{p-1}$ and $N(fg) = (Nf)g + f(Ng)$. (The intuition is that ℓ is "$\log T$".) We can also extend ι_n to

$$\begin{aligned} \iota_n : R_A(\pi_K)[\ell] &\to (A \otimes K_n)[\![\tilde{t}]\!] \\ \ell &\mapsto \log(\zeta_{p^n} \exp(\tilde{t}/p^n) - 1). \end{aligned}$$

8.3 φ-modules

Definition 8.3.1. By a φ-module over $R_A^\alpha(\pi_K)$, we mean a finite projective (hence coadmissible) module M over $R_A^\alpha(\pi_K)$ together with a map

$$\varphi : M \to M_{\alpha^{1/p}} := M \otimes_{R_A^\alpha(\pi_K)} R_A^{\alpha^{1/p}}(\pi_K)$$

which is φ-semilinear and induces an isomorphism

$$\left(\varphi^* M := M \otimes_{R_A^\alpha(\pi_K),\varphi} R_A^{\alpha^{1/p}}(\pi_K)\right) \xrightarrow{\sim} M_{\alpha^{1/p}}.$$

Lemma 8.3.2. *This is the same as a vector bundle $\mathscr{M}$ over $R_A^\alpha(\pi_K)$ (coming from $\mathscr{M}_\gamma$ over $R_A^{[\alpha,\gamma]}(\pi_K)$ for all γ) together with an isomorphism*

$$\varphi : \varphi^* \mathscr{M}_\gamma \xrightarrow{\sim} \mathscr{M}_{\gamma^{1/p},[\alpha^{1/p},\gamma^{1/p}]} := \mathscr{M}_{\gamma^{1/p}} \otimes_{R_A^{[\alpha,\gamma^{1/p}]}(\pi_K)} R_A^{[\alpha^{1/p},\gamma^{1/p}]}(\pi_K).$$

A warning about the proof, which we won't do because it's standard: $\Gamma(\mathscr{M})$ is finitely generated and projective because of the φ structure. Without it, the number of generators might blow up as you get near the boundary, but the φ structure moves things closer to the boundary while maintaining uniformity in the number of generators.

Definition 8.3.3. By a φ-module over $R_A(\pi_K)$, we mean a finite projective module M over $R_A(\pi_K)$ together with $\varphi : \varphi^* M \xrightarrow{\sim} M$ arising by base change from some φ-module M_α over $R_A^\alpha(\pi_K)$ for some $\alpha < 1$.

Lemma 8.3.4. *Let $L/\mathbb{Q}_p$ be finite, $M/R_L(\pi_K)$ be finite projective, and $\varphi : M \to M$ be φ-semi-linear. Suppose $M = \langle \varphi M \rangle_{R_L(\pi_K)}$ (note that φM might not itself be a module over $R_L(\pi_K)$, due to semilinearity). Then for any α sufficiently close to 1, there is a unique $R_L^\alpha(\pi_K)$-submodule $M^\alpha \subset M$ such that*

1. *$M^\alpha \otimes_{R_L^\alpha(\pi_K)} R_L(\pi_K) \xrightarrow{\sim} M$, and*
2. *$M^\alpha \otimes_{R_L^\alpha(\pi_K)} R_L^{\alpha^{1/p}}(\pi_K)$ has a basis contained in φM^α (a finite-level version of the hypothesis $M = \langle \varphi M \rangle_{R_L(\pi_K)}$; basically "$\varphi$ is surjective", except it's not a module map).*

Furthermore, (M, φ) is a φ-module arising from this M^α. If $\beta > \alpha$, then

$$M^\alpha \otimes_{R_L^\alpha(\pi_K)} R_L^\beta(\pi_K) \xrightarrow{\sim} M^\beta.$$

If $M/R_A(\pi_K)$ is a φ-module then for $n \gg 0$ let

$$M_n = (A \otimes K_n)[\![\tilde{t}]\!] \otimes_{R_A^\alpha(\pi_K)} M^\alpha.$$

We have an isomorphism

$$\begin{aligned} A \otimes K_{n+1}[\![\tilde{t}]\!] \otimes_{A \otimes K_n[\![\tilde{t}]\!]} M_n &\xrightarrow{\sim} M_{n+1} \\ 1 \otimes (f \otimes m) &\mapsto f \otimes \varphi m. \end{aligned}$$

Lemma 8.3.5. *Suppose $L/\mathbb{Q}_p$ is finite. (This lemma really doesn't work over general affinoids!) Suppose also that $M_0/R_L(\pi_K)$ is a φ-module. Then there is an equivalence of categories between*

$\{\varphi$-modules M with an embedding $M \hookrightarrow M_0[1/t]$ inducing $M[1/t] \xrightarrow{\sim} M_0[1/t]\}$ and

$\{$collections $\{M_n\}$ of compatible free $(L \otimes K_n)[\![\tilde{t}]\!]$-modules for $n \gg 0$ (compatible meaning

$$M_n \otimes_{(L\otimes K_n)[\![\tilde{t}]\!]} (L \otimes K_{n+1})[\![\tilde{t}]\!] \xrightarrow{\sim} M_{n+1})$$

together with compatible $M_n \hookrightarrow M_{0,n}[1/\tilde{t}]$ inducing $M_n[1/\tilde{t}] \xrightarrow{\sim} M_{0,n}[1/\tilde{t}]\}$.

That is, given one φ-module M_0, we can generate a bunch of other ones by perturbing at a p^nth root of unity. If we have an M as in the first category, it can only differ from M_0 at the roots of t, i.e. the p-power roots of unity, so we need to say what it is at the p-power roots of unity. The data in the second category tells you this, and since everything has to be compatible, it is really the data for one p^nth root of unity for sufficiently large n, which can be translated to everything else via Frobenius. Actually, this is the only way to change M_0 by a finite amount, since if you wanted to change it at a member of an infinite orbit you'd have to change it at every member of the orbit.

8.4 (φ, Γ)-modules

Definition 8.4.1. By a (φ, Γ)-module $M/R_A^\alpha(\pi_K)$, we mean a φ-module (φ, M) together with a continuous semilinear action of Γ_K which commutes with φ.

By a (φ, Γ)-module $M/R_A(\pi_K)$, we mean a φ-module (φ, M) with a continuous semilinear action of Γ_K which arises from a (φ, Γ)-module $M^\alpha/R_A^\alpha(\pi_K)$ for some $\alpha < 1$.

Lemma 8.4.2. *If $L/\mathbb{Q}_p$ is finite and $M/R_L(\pi_K)$ is a finite projective module with commuting semilinear actions of φ and Γ_K (the latter continuous), and $\langle \varphi M\rangle_{R_L(\pi_K)} = M$, then M is a (φ, Γ)-module.*

Theorem 8.4.3. *There exists an exact, fully faithful tensor functor M_{rig} from*

$\{$finite projective A-modules with a continuous A-linear action of $G_K\}$ to

$\{(\varphi, \Gamma)$-modules over $R_A(\pi_K)\}$

(which is not *essentially surjective). It has the following properties:*

1. *it is compatible with base change in A.*
2. $\mathrm{rank}_{R_A(\pi_K)} M_{rig}(V) = \mathrm{rank}_A(V)$.
3. *if K'/K is finite, then $M_{rig}(V|_{G_{K'}}) \cong M_{rig}(V) \otimes_{R_A(\pi_K)} R_A(\pi_{K'})$ as $(\varphi, \Gamma_{K'})$-modules.*

Lemma 8.4.4. *Suppose K'/K is finite Galois. Then there is an equivalence of categories between*

$\{(\varphi, \Gamma_K)$-modules over $R_A(\pi_K)\}$ and

$\{(\varphi, \Gamma_{K'})$-modules over $R_A(\pi_{K'})$ with a semi-linear action of $\mathrm{Gal}(K'_\infty/K)$ extending the action of $\Gamma_{K'} \subset \mathrm{Gal}(K'_\infty/K)\}$

taking $M^{H_K} \hookleftarrow M$ and $N \mapsto N \otimes_{R_A(\pi_K)} R_A(\pi_{K'})$.

The extension $R_A(\pi_K)[\ell]$ we introduced will be used for example to detect when a representation is semistable. It will be used in going from filtered WD reps to (φ, Γ)-modules.

9 February 11: $D_{sen,dR,cris,st}$.

9.1 Recap and loose ends

Recall that we defined a (φ, Γ)-module as a finite projective module M over $R_A^\alpha(\pi_K)$, along with $\varphi : \varphi^* M \xrightarrow{\sim} M|_{\mathrm{Sp}\, R_A^{\alpha^{1/p}}(\pi_K)}$, and a semilinear continuous action of Γ_K, where we are given $\varphi : R_A^\alpha(\pi_K) \to R_A^{\alpha^{1/p}}(\pi_K)$ and an action of $\Gamma_K = \mathrm{Gal}(K_\infty/K)$ on $R_A^\alpha(\pi_K)$. If $R_A(\pi_K) = \bigcup R_A^\alpha(\pi_K)$, an M over $R_A(\pi_K)$ will come from a (φ, Γ_K)-module over some $R_A^\alpha(\pi_K)$.

If K'/K is finite and M is a $(\varphi, \Gamma_{K'})$-module over $R_A(\pi_{K'})$, we may define

$$\mathrm{Ind}_{K'}^{K} M = \mathrm{Ind}_{\Gamma_{K'}}^{\Gamma_K} M$$

viewed as a $R_A(\pi_K)$-module; this is a (φ, Γ_K)-module over $R_A(\pi_K)$.

Last time, we stated the following theorem.

Theorem 9.1.1. *There is an exact fully faithful tensor functor M_{rig} from $\{$finite projective A-modules with a continuous A-linear action of $G_K\}$ to $\{(\varphi, \Gamma_K)$-modules over $R_A(\pi_K)\}$ with the properties that*

- *it commutes with change of the coefficients A.*
- *if K'/K is finite and V is given over G_K, we have*

$$M_{rig}(V|_{G_{K'}}) \cong M_{rig}(V) \otimes_{R_A(\pi_K)} R_A(\pi_{K'}).$$

- *if K'/K is finite and V is given over $G_{K'}$, we have*

$$M_{rig}\left(\mathrm{Ind}_{G_{K'}}^{G_K} V\right) \cong \mathrm{Ind}_{K'}^{K} M_{rig}(V).$$

- $\mathrm{rank}_{R_A(\pi_K)} M_{rig}(V) = \mathrm{rank}_A(V)$.

9.2 Differentials and Sen modules

Suppose M is a (φ, Γ_K)-module. We can define

$$\nabla_M = \frac{\log \gamma}{\log \epsilon_p(\gamma)} : M \to M$$

for any $\gamma \in \Gamma_K \setminus \{1\}$ sufficiently close to 1 (so that $\log \gamma$ is defined). This satisfies

$$\nabla_M(fm) = (\nabla f)m + f(\nabla_M m)$$

for all $m \in M$ and $f \in R_A(\pi_K)$, where ∇ is the differential we defined on the Robba ring. We can also define

$$M_n = M \otimes_{R_A^\alpha(\pi_K), \iota_n} (A \otimes K_n)[\![\tilde{t}]\!]$$

for $n \gg 0$ (for α large enough that M exists over it). This has an action of Γ_K and ∇_{M_n}. We have

$$\nabla_{M_n}(fm) = \left(\tilde{t}\frac{d}{d\tilde{t}}f\right) m + f\nabla_{M_n}(m).$$

This means that $\nabla_{M_n}\tilde{t}M_n \subset \tilde{t}M_n$, hence ∇_{M_n} induces an $A \otimes K_n$-linear endomorphism of $M_n/\tilde{t}M_n$. The Sen module is

$$D^{\circ}_{sen}(M) = (M_n/\tilde{t}M_n)^{\Gamma_K}$$

over $A \otimes K$; this is independent of n for $n \gg 0$ (since going from n to $n+1$ just means extending scalars from K_n to K_{n+1}). We write

$$\Theta_{sen} = -\nabla_{M_n}$$

for the Sen operator acting on $D^{\circ}_{sen}(M)$. We have

$$\operatorname{rank}_A D^{\circ}_{sen}(M) = [K : \mathbb{Q}_p] \operatorname{rank}_{R_A(\pi_K)} M.$$

Let $L/\mathbb{Q}_p$ be finite (Richard thinks the following should also work for an affinoid algebra A, but can't find a reference). Suppose V/L has a continuous action of G_K. Recall that $\Gamma_K = G_K/H_K$. We have a $\widehat{K}_\infty$-module

$$(V \otimes_{\mathbb{Q}_p} \widehat{\overline{\mathbb{Q}}}_p)^{H_K}$$

(where H_K acts diagonally on both factors) which by Hilbert's theorem 90 satisfies

$$(V \otimes_{\mathbb{Q}_p} \widehat{\overline{\mathbb{Q}}}_p)^{H_K} \otimes_{\widehat{K}_\infty} \widehat{\overline{\mathbb{Q}}}_p \cong V \otimes_{\mathbb{Q}_p} \widehat{\overline{\mathbb{Q}}}_p.$$

This contains $D_{sen}(V)$, the union of all finite-dimensional Γ_K-invariant K_∞-subspaces. This is (evidently) stabilized by the Γ_K-action, and it turns out we have an isomorphism

$$D_{sen}(V) \otimes_{K_\infty} \widehat{K}_\infty \xrightarrow{\sim} (V \otimes_{\mathbb{Q}_p} \widehat{\overline{\mathbb{Q}}}_p)^{H_K}.$$

$D_{sen}(V)$ is again acted on by the operator $\Theta_{sen} = -\frac{\log \gamma}{\log \epsilon_p(\gamma)}$ for $\gamma \in \Gamma_K \setminus \{1\}$ sufficiently close to 1, and this action commutes with the Γ_K-action since Γ_K is abelian.

Proposition 9.2.1. *We have $D^{\circ}_{sen}(M_{rig}(V)) \otimes_K K_\infty = D_{sen}(V)$, and the two Θ_{sen}s agree.*

Let $\operatorname{char}_{\Theta_{sen}}(x) \in (L \otimes K_\infty)[x]$ be the characteristic polynomial; since Θ_{sen} commutes with the Γ_K-action, $\operatorname{char}_{\Theta_{sen}}(x)$ is invariant under the Γ_K-action and hence actually lies in $(L \otimes K)[x]$. If $\tau : K \hookrightarrow L$, we write $HTS_\tau(V)$ (Hodge-Tate-Sen weights) for the roots of $(1 \otimes \tau) \operatorname{char}_{\Theta_{sen}}(x) \in L[x]$ with multiplicity, which lie in $\overline{L}$. If V is de Rham, then $HTS_\tau(V) = HT_\tau(V)$ (in particular $HTS_\tau(V)$ consists of integers), and Θ_{sen} is semisimple. (Note that the Sen operator Θ_{sen} contains more information than $HTS_\tau(V)$ if it is not semisimple.)

9.3 D_{dR}

Let $L/\mathbb{Q}_p$ be finite (for real this time), and M an $R_L(\pi_K)$-module. Write

$$D_{dR}(M) = M_n[1/\tilde{t}]^{\Gamma_K}$$

for $n \gg 0$, which is a module over $L \otimes K$ independent of n. We have

$$M_n[1/\tilde{t}]^{\nabla_{M_n}=0} = D_{dR}(M) \otimes_K K_n$$

for $n \gg 0$, because Γ_K acts through a finite quotient on the LHS and base changing from K to K_n gives the same thing by Hilbert's theorem 90. We have a natural embedding

$$D_{dR}(M) \otimes_{L\otimes K} (L \otimes K_n)((\tilde{t})) \hookrightarrow M_n[1/\tilde{t}]$$

so

$$\dim_K D_{dR}(M) \leq \dim_{K_n((\tilde{t}))} M_n[1/\tilde{t}].$$

We call M de Rham if the dimensions are equal, or equivalently if the embedding is an isomorphism. (Some people, in the context of p-adic differential equations, also call this "locally trivial": "there is a full set of solutions to the differential equation $\nabla_{M_n} = 0$ locally near a root of unity".)

The differential

$$\nabla_{M_n} : D_{dR}(M) \otimes_K K_n[\![\tilde{t}]\!] \to \tilde{t} D_{dR}(M) \otimes_K K_n[\![\tilde{t}]\!]$$

has image divisible by $\tilde{t}$, so if we were working with this lattice it would look like the HTS numbers were 0, but this is not our original lattice M_n, which is where the HTS numbers are actually defined.

We define

$$\mathrm{Fil}^i D_{dR}(M) = (\tilde{t}^i M_n)^{\Gamma_K}$$

for $n \gg 0$. This is a decreasing filtration which is (0) if $i \gg 0$ and $D_{dR}(M)$ if $i \ll 0$.

Proposition 9.3.1. *$D_{dR}(M_{rig}(V)) = D_{dR}(V)$.*

If M is de Rham and we choose $\tau : K \hookrightarrow L$, then $HTS_\tau(M)$ contains i with multiplicity $\dim_L \mathrm{gr}^i D_{dR}(M) \otimes_{K\otimes L, \tau\otimes 1} L$.

Example 9.3.2. The proof of this if $K = \mathbb{Q}_p$ (the argument is exactly the same in general, just more notation): choose a basis e_i of $D_{dR}(M)$ over L compatible with the filtration. That means for all j, $\mathrm{Fil}^j D_{dR}(M) = \langle e_1, \ldots, e_{d(j)} \rangle$. Say $e_i \in \mathrm{Fil}^{d(i)} \setminus \mathrm{Fil}^{d(i)+1}$. Then e_i lies in $\tilde{t}^{d(i)} M_n$, $\tilde{t}^{-d(i)} e_i$ lies in M_n, and the $\tilde{t}^{-d(i)} e_i$ form a basis of M_n over $(L \otimes \mathbb{Q}_{p,n})[\![t]\!]$.

Then these also give a basis of $M_n/\tilde{t}M_n$. We have

$$-\nabla_{M_n}(\tilde{t}^{-d(i)} e_i) = d(i)\tilde{t}^{-d(i)} e_i - \tilde{t}^{-d(i)} \nabla_{M_n} e_i$$

and if M is de Rham then the second term is 0, so

$$\Theta_{sen}(\tilde{t}^{-d(i)} e_i) = d(i)(\tilde{t}^{-d(i)} e_i)$$

which is to say that $\tilde{t}^{-d(i)} e_i$ is an eigenvector of Θ_{sen} with eigenvalue $d(i)$, and therefore $HTS(M) = \{d(i)\}_i$.

Lemma 9.3.3. *Suppose that $0 \to M_1 \to M \to M_2 \to 0$ is an exact sequence of (φ, Γ_K)-modules over $R_L(\pi_K)$ with M_1 and M_2 de Rham. Suppose also that for all τ, every element of $HT_\tau(M_1)$ is strictly less than every element of $HT_\tau(M_2)$. Then M is de Rham.*

Proof. Choose n sufficiently large so that $D_{dR}(M_i) \otimes_K K_n[\![\tilde{t}]\!] \xrightarrow{\sim} M_{i,n}$ for $i = 1, 2$. We get a left-exact sequence

$$0 \to D_{dR}(M_1) \to D_{dR}(M) \to D_{dR}(M_2)$$

and we need to show it is short exact (the rightmost map is surjective) because that will show that $D_{dR}(M)$ has the right rank. WLOG L contains τK for all $\tau : K \hookrightarrow \overline{L}$, so it suffices to prove that

$$D_{dR,\tau}(M) := D_{dR}(M) \otimes_{L \otimes K, 1 \otimes \tau} L \twoheadrightarrow D_{dR,\tau}(M_2)$$

for all τ. Let $m = \max HT_\tau(M_1) + 1$; this is $\leq$ every element of $HT_\tau(M_2)$. We are going to prove that

$$\mathrm{Fil}^m D_{dR,\tau}(M) \twoheadrightarrow \mathrm{Fil}^m D_{dR,\tau}(M_2) = D_{dR,\tau}(M_2).$$

Write $L \otimes_{\mathbb{Q}_p} K_n = \prod L_i$, and let L_0/K_n be a factor such that $K \hookrightarrow K_n \to L_0$ commutes with $K \hookrightarrow^\tau L \to L_0$. Tensor both sides by $\otimes_L L_0$. We have

$$\mathrm{Fil}^m D_{dR,\tau}(M) \otimes_L L_0 = (\tilde{t}^m M_n \otimes_{L \otimes K_n} L_0)^{\Gamma_{K_n}}$$
$$\mathrm{Fil}^m D_{dR,\tau}(M_2) \otimes_L L_0 = (\tilde{t}^m M_{2,n} \otimes_{L \otimes K_n} L_0)^{\Gamma_{K_n}}$$

because (as we said before, by Hilbert 90)

$$D_{dR}(M) \otimes_K K_n = M_n[1/\tilde{t}]^{\Gamma_{K_n}}$$

and so

$$D_{dR}(M) \otimes_{L \otimes K} L_0 = (D_{dR}(M) \otimes_K K_n) \otimes_{L \otimes K_n} L_0 = M_n[1/\tilde{t}]^{\Gamma_{K_n}} \otimes_{L \otimes K_n} L_0.$$

(Here $\Gamma_{K_n} = \mathrm{Gal}(K_\infty/K_n)$). Let $\Gamma_{K_n} = \langle \gamma \rangle$. We have a short exact sequence

$$0 \to \tilde{t}^m M_{1,n} \otimes_{K_n \otimes L} L_0 \to \tilde{t}^m M_n \otimes_{K_n \otimes L} L_0 \to \tilde{t}^m M_{2,n} \otimes_{K_n \otimes L} L_0 \to 0.$$

Apply $\gamma - 1$ to each term and use the snake lemma on the resulting map of SESs to get

$$0 \to (\tilde{t}^m M_{1,n} \otimes L_0)^{\Gamma_{K_n}} \to (\tilde{t}^m M_n \otimes L_0)^{\Gamma_{K_n}} \to (\tilde{t}^m M_{2,n} \otimes L_0)^{\Gamma_{K_n}} \to \tilde{t}^m M_{1,n} \otimes L_0/(\gamma - 1)$$

(where the first three terms are the respective kernels of $\gamma - 1$). It suffices to prove that $\gamma - 1$ is an isomorphism on $\tilde{t}^m M_{1,n} \otimes L_0$ (so in fact both the first and the last terms are 0, though we only need the last).

As in our previous example calculating HTS numbers, choose a basis $\{e_i\}$ of $D_{dR,\tau}(M_1)$ compatible with the filtration, with $e_i \in (\mathrm{Fil}^{d(i)} \setminus \mathrm{Fil}^{d(i)+1}) D_{dR,\tau}(M_1)$. We know that $\{\tilde{t}^{-d(i)} e_i\}$ is a basis of $M_{1,n} \otimes L_0$ over $L_0[\![\tilde{t}]\!]$, hence $\{\tilde{t}^{m-d(i)} e_i\}$ is a basis of $\tilde{t}^m M_{1,n} \otimes L_0$ over $L_0[\![\tilde{t}]\!]$ (note that $m - d(i) > 0$ since $m > d_i \in HT_\tau(M_1)$). So any element of $\tilde{t}^m M_1 \otimes_{K_n \otimes L} L_0$ can be written uniquely as

$$\sum_i e_i \sum_{j=m-d(i)}^{\infty} f_{ij} \tilde{t}^j.$$

If we apply $\gamma - 1$ to this, since γ fixes e_i and passes through the coefficient field elements $f_{ij} \in L_0$, we get

$$\sum_i e_i \sum_{j=m-d(i)}^{\infty} f_{ij}(\epsilon_p(\gamma)^j - 1)\tilde{t}^j.$$

Since $j = m - d(i) > 0$ for all j appearing in the sum, we have $\epsilon_p(\gamma)^j - 1 \neq 0$, so indeed

$$\gamma - 1 : \tilde{t}^m M_1 \otimes_{K_n \otimes L} L_0 \xrightarrow{\sim} \tilde{t}^m M_1 \otimes_{K_n \otimes L} L_0.$$

□

Exercise 9.3.4. Write this out in the case $K = \mathbb{Q}_p$ (so $L_0 = L$, etc.), where there's less notation, to help see what's going on.

9.4 D_{cris}, D_{st}, etc.

There are also analogues of the other functors of p-adic Hodge theory. The definition of D_{cris} is a global definition over the whole annulus, unlike D_{dR} which operates locally at a root of unity. Define

$$D_{cris}(M) = M[1/t]^{\Gamma_K}$$

a $K_0 \otimes A$-module with a semilinear Frobenius. Similarly, define

$$D_{st}(M) = (M \otimes_{R_A(\pi_K)} R_A(\pi_K)[\ell, 1/t])^{\Gamma_K}$$

a $K_0 \otimes A$-module with a semilinear φ and an operator N such that $\varphi N = pN\varphi$, i.e. a (φ, N)-module. If K'/K is finite Galois, define

$$D_{K'-st}(M) = (M \otimes_{R_A(\pi_K)} R_A(\pi_{K'})[\ell, 1/t])^{\Gamma_{K'}}$$

a $(\varphi, N, \mathrm{Gal}(K'/K))$-module with coefficients in A, and

$$D_{pst}(M) = \varinjlim_{K'} D_{K'-st}(M)$$

a (φ, N, G_K)-module with coefficients in A. We have

$$\mathrm{rank}_{K_0'/K_0/K_0} D_{K'-st/st/cris}(M) \leq \mathrm{rank}_{R_A(\pi_K)} M$$

and if these are equal we say M is K'-semistable/semistable/crystalline. We have

$$D_\bullet(M_{rig}(V)) = D_\bullet(V)$$

for $\bullet = cris, st, K' - st$.

Next time, we'll talk about the following theorem. In p-adic Hodge theory, de Rham implies potentially semistable. This is proved by proving it for (φ, Γ)-modules and then descending to Galois representations. For (φ, Γ)-modules, one thinks of the de Rham condition as saying that a certain differential equation has a full set of solutions formally at a root of unity, and the crystalline/(potentially) semistable conditions as saying that it has a full set of solutions over the whole annulus. The big theorem is that given the Frobenius structure, being able to solve an equation locally at a root of unity means you can solve it globally.

10 February 16: M_{rig} and rank 1 (φ, Γ)-modules.

10.1 Recap

Last time, we were working with M, a (φ, Γ_K)-module over $R_L(\pi_K)$. We looked at its expansions around roots of unity

$$M_n = M \otimes_{R_L(\pi_K),\iota_n} (L \otimes K_n)[\![\tilde{t}]\!]$$

for $n \gg 0$ (which all carry the same data because the Frobenius takes p-power roots of unity to previous ones). We defined

$$D_{dR}(M) = M_n[1/\tilde{t}]^{\Gamma_K},$$

a module over $L \otimes K$ (which no longer has the action of Frobenius), and by differentiating the action of Γ_K, we got an operator ∇_{M_n}, and saw that

$$M_n[1/\tilde{t}]^{\nabla_{M_n}=0} = D_{dR}(M) \otimes_K K_n$$

(once the infinitesimal action of Γ_K is trivial, Γ_K acts through a finite quotient, at which point Hilbert's Theorem 90 gives you the K-structure of the K_n-vector space). We got an embedding

$$D_{dR}(M) \otimes_K K_n((\tilde{t})) \hookrightarrow M_n[1/\tilde{t}]$$

and called M de Rham if this was an isomorphism. We also defined

$$D_{cris}(M) = M[1/t]^{\Gamma_K}$$

with φ, over $K_0 \otimes L$,

$$D_{st}(M) = M[\ell, 1/t]$$

with φ, N, over $K_0 \otimes L$, and

$$D_{K'-st}(M) = (M \otimes R_L(\pi'_K)[1/t, \ell])^{\Gamma_{K'}}$$

with $\varphi, N, \mathrm{Gal}(K'/K)$, over $K'_0 \otimes L$. Finally

$$D_{pst} = \varinjlim_{K'} D_{K'-st}.$$

These are analogues of the similarly-named modules associated to Galois representations, and recover them when M comes from a Galois representation.

10.2 M_{rig}

If M is de Rham, there is a finite extension K'/K and a finite projective $(\varphi, \Gamma_{K'})$-module $M_0 \subset R_L(\pi'_K)[1/t] \otimes M$ such that

- $M_0[1/t] = R_L(\pi_{K'})[1/t] \otimes M$,
- $\nabla_M M_0 \subset t M_0$,

- $D_{K'-st}(M) = M_0[\ell]^{\Gamma_{K'}}$,
- $D_{K'-st}(M) \otimes_{K'_0} K'_n \xrightarrow{\sim} D_{dR}(M) \otimes_K K'_n$,
- $M_0[\ell] \cong D_{K'-st}(M) \otimes_{L\otimes K'_0} R_L(\pi_{K'})[\ell]$.

(That is to say, $D_{K'-st}(M)$ has maximal rank; M being de Rham means there's a full set of solutions to the p-adic differential equation $\nabla = 0$ in a neighborhood of a root of unity; these solutions extend to the whole annulus.)

In fact, for $L/\mathbb{Q}_p$ finite, we get an equivalence of categories from

{de Rham (φ, Γ_K)-modules over $R_L(\pi_K)$} to

{filtered (φ, N, G_K)-modules with coefficients in L}

as follows. If D is in the latter category, if $N = 0$, we could define an associated (φ, Γ_K)-module by

$$R_L(\pi_K) \otimes_{K_0 \otimes L} D$$

with action of $\gamma \in \Gamma_K$ given by $\gamma \otimes 1$ and φ given by $\varphi \otimes \varphi$; this has local expansion

$$K_n[\![\tilde{t}]\!] \otimes_{\mathrm{Frob}_p^n, K_0} D.$$

For general N, we could define an associated (φ, Γ_K)-module by

$$M_0 = (R_L(\pi_K)[\ell] \otimes_{K_0 \otimes L} D)^{N=0}$$

(by N we mean $N \otimes 1 + 1 \otimes N$), which is indeed a (projective) (φ, Γ_K)-module; this has local expansion

$$M_{0,n} = (K_n[\![\tilde{t}]\!] \otimes_{\mathrm{Frob}_p^n, K_0} D)^{N=0}.$$

This M_0 is the "second lattice" with nice properties associated to $M_{rig}(D)$ from the beginning of this subsection; it doesn't have the information from the filtration of D yet. In order to identify $M_{rig}(D) \subset M_0[1/t]$, by Lemma 8.3.5, we need to choose sublattices of $M_{0,n}[1/\tilde{t}]$ for each n. More precisely, we want a sublattice of

$$\begin{aligned} K_n((\tilde{t})) \otimes_{K_n[\![\tilde{t}]\!]} M_{0,n} &= (K_n((\tilde{t})) \otimes_{\mathrm{Frob}_p^n, K_0} D)^{N=0} \\ &\xleftarrow{\sim, 1\otimes\varphi^n} (K_n((\tilde{t})) \otimes_{K_0} D)^{N=0} \\ &= (K_n((\tilde{t})) \otimes_K D_K)^{N=0} \end{aligned}$$

where $D_K = D \otimes_{K_0} K$ (this is the part with the filtration on it), and our desired sublattice of $(K_n((\tilde{t})) \otimes_K D_K)^{N=0}$ is

$$\mathrm{Fil}^0 (K_n((\tilde{t})) \otimes_K D_K)^{N=0} = \left(\sum_i \tilde{t}^{-i} K_n[\![\tilde{t}]\!] \otimes_K \mathrm{Fil}^i D_K \right)^{N=0}.$$

(That is, this is our choice of $M_{rig}(D)_n$, which depends on the filtration on D.)

We have $M_{rig}(D) = M_{rig}(V)$ for some V if and only if D is admissible. (Recall that this is a "Newton polygon lies above Hodge polygon" condition comparing the valuations of the eigenvalues of Frobenius with the jumps in the filtration.) As a result, if V is de Rham, then $M_{rig}(V)$ is de Rham, so $M_{rig}(V) = M_{rig}(D)$ for some D (this is the hard part coming from solving the differential equation), and D is admissible. Therefore $D_{pst}(V) = D$ and V is potentially semistable.

10.3 Cohomology and globalization

You can recover Galois cohomology on the level of (φ, Γ)-modules. Let M over $R_A(\pi_K)$ be a (φ, Γ_K)-module, and $\Delta_K \subset \Gamma_K$ be the maximal p-power torsion subgroup (i.e. $\{1\}$ if $p > 2$ or $\{\pm 1\}$ if $p = 2$). Let $\Gamma_K/\Delta_K = \langle \gamma_K \rangle$. Define $H^\bullet_{\varphi,\gamma_K}(M)$ to be the cohomology of

$$M^{\Delta_K} \xrightarrow{(\varphi-1,\gamma_K-1)} M^{\Delta_K} \oplus M^{\Delta_K} \xrightarrow{(\gamma_K-1)+(1-\varphi)} M^{\Delta_K}.$$

Proposition 10.3.1. *1. $H^\bullet_{\varphi,\gamma_K}(M)$ is a finitely generated A-module.*

2. $H^\bullet_{\varphi,\gamma_K}(M_{rig}(V)) \cong H^\bullet_{cts}(G_K, V)$ if V is a projective A-module with continuous G_K-action.

Also define

$$\tilde{H}(M) = \{m \in M^{\Delta_K} \mid \text{ there is } n \in \mathbb{Z}_{>0} \text{ such that } (\varphi - 1)^n m = (\gamma_K - 1)^n m = 0\}.$$

Then $\tilde{H}$ is left exact, and $\tilde{H}(M) \neq (0)$ if and only if $H^0_{\varphi,\gamma_K}(M) \neq (0)$ (by successively replacing m by $(\varphi - 1)m$ or $(\gamma_K - 1)m$), also if and only if $\mathrm{Hom}(R_A(\pi_K), M) \neq (0)$ (because such a homomorphism is determined by where 1 goes to, and the possible images of 1 are exactly the elements of $H^0_{\varphi,\gamma_K}(M) \neq (0)$). Recall that we explicitly calculated

$$\tilde{H}(R_L(\pi_K)) = L \otimes_{\mathbb{Q}_p} K_0' = H^0_{\varphi,\gamma_K}(R_L(\pi_K)).$$

If $X/\mathbb{Q}_p$ is a rigid space, we can define a (φ, Γ_K)-module over $R^\alpha_X(\pi_K)$ to be a coherent sheaf of modules $\mathscr{M}$ over $X \times_{\mathbb{Q}_p} \Delta[\alpha, 1)$together with

$$\varphi : (1 \times \varphi)^* \mathscr{M} \xrightarrow{\sim} \mathscr{M}$$

over $X \times \Delta[\alpha^{1/p}, 1)$and a continuous $\mathscr{O}_X$-linear, $R^\alpha_{\mathbb{Q}_p}(\pi_K)$-semilinear Γ_K-action such that for all affinoid subdomains $\mathrm{Sp}(A) \subset X$, $\mathscr{M}(\mathrm{Sp}(A))$ is a (φ, Γ_K)-module over $R^\alpha_A(\pi_K)$.

Alternatively, we can define $R^\alpha_X(\pi_K)$ as a sheaf on X and think of $\mathscr{M}$ as a sheaf on X. We can also define $R_X(\pi_K)$-modules, in which case this alternative definition doesn't change, but the previous one might have to be modified.

10.4 Examples of rank 1 (φ, Γ_K)-modules

These will be important due to being easy to classify.

1. Let $\hat{\delta} : G_K \to A^\times$ be continuous. Then we get a rank 1 (φ, Γ_K)-module $M_{rig}(\hat{\delta})$, and since M_{rig} is a tensor functor, we have $M_{rig}(\hat{\delta}_1\hat{\delta}_2) = M_{rig}(\hat{\delta}_1) \otimes M_{rig}(\hat{\delta}_2)$.

2. Let $a \in A^\times$. We will define a (φ, Γ_K)-module M_a of rank 1. Define D_a over $A \otimes_{\mathbb{Q}_p} K_0$ as follows. First define
$$\tilde{D} = \bigoplus_{i=1}^{[K_0:\mathbb{Q}_p]} (A \otimes_{\mathbb{Q}_p} K_0) e_i$$

with Frobenius

$$\varphi e_i = \begin{cases} e_{i-1} & \text{if } i > 1 \\ a e_{[K_0:\mathbb{Q}_p]} & \text{if } i = 1. \end{cases}$$

This is $A \otimes_{\mathbb{Q}_p} K_0$-linear for the trivial $A \otimes K_0$-action. Define a second $A \otimes_{\mathbb{Q}_p} K_0$ action "." by

$$(\alpha \otimes \beta).e_i = \alpha \otimes (\mathrm{Frob}_p^{-i} \beta) e_i$$

so that φ is $1 \otimes \mathrm{Frob}_p^{-1}$-semilinear WRT the "." action of $A \otimes K_0$. Now define D_a to be the submodule of $\tilde{D}$ fixed by $1 \otimes \sigma$ for all $\sigma \in \mathrm{Gal}(K_0/\mathbb{Q}_p)$. D_a still has an action of φ and inherits the "." action of $A \otimes K_0$, and is free of rank 1 over $A \otimes K_0$. Finally define

$$M_a = R_A(\pi_K) \otimes_{A \otimes K_0} D_a.$$

This is free of rank 1 over $R_A(\pi_K)$, with γ acting as $\gamma \otimes 1$ and φ acting as $\varphi \otimes \varphi$, so gives a (φ, Γ_K)-module. (This is the same as what we were doing before in general, but here we don't have to deal with N and the filtration.) We have

(a) $M_a \otimes M_b = M_{ab}$.

(b) if $A = L$, $\mathrm{Fil}^0 D_a = D_a$, and $\mathrm{Fil}^1 D_a = (0)$, then $M_a = M_{rig}(D_a)$.

(c) if $\hat{\delta} : G_K/I_K \to A^\times$ is continuous, then $M_{rig}(\hat{\delta}) = M_{\hat{\delta}(\mathrm{Frob}_K)}$. (This is true over a general affinoid A.)

Here recall that if V/A is a finite projective module with a continuous action of G_K, M_{rig} as a functor from

{finite projective A-modules with continuous action of G_K} to

$\{(\varphi, \Gamma_K)$-modules over $R_A(\pi_K)\}$

is a fully faithful exact tensor functor which we mentioned but didn't define. It is given by $M_{rig}(V) = (B_A \otimes V)^{H_K}$, where we don't want to define B_A, but the important things to remember about it are that $B^{H_K} = R_A(\pi_K)$ and that B_A has actions of φ and G_K, so that $M_{rig}(V)$ has actions of φ and Γ_K.

Separately, we have the second M_{rig} from

{filtered (φ, N, G_K)-modules}

(which are equivalent over large coefficients to filtered WD reps of W_K) to

$\{(\varphi, \Gamma_K)$-modules over $R_A(\pi_K)\}$, which we defined earlier in this lecture.

If V is de Rham, then $M_{rig}(V) = M_{rig}(D_{pst}(V))$, but the second functor is more general, existing without any admissibility assumptions. (The compatibility $M_{rig}(\hat{\delta}) = M_{\hat{\delta}(\mathrm{Frob}_K)}$ should follow from this.)

3. If $\delta : W_K \to A^\times$ is a continuous character (so it's continuous on inertia and Frobenius can go to anything), we can write $\delta = \delta_1 \delta_2$ with δ_1 unramified and δ_2 extending to

$\hat{\delta}_2 : G_K \to A^\times$. For example, if $\varphi \in W_K$ is a lift of Frobenius, we can set $\delta_1 : W_K \to W_K/I_K \to A^\times$ to be $\mathrm{Frob}_K \mapsto \delta(\varphi)$; then $\delta_2 = \delta\delta_1^{-1} : \varphi \mapsto 1$ extends to G_K. Then let

$$R_A(\pi_K)(\delta) = M_{rig}(\hat{\delta}_2) \otimes M_{\delta_1(\mathrm{Frob}_K)};$$

this is well-defined independently of the choice of δ_1, δ_2 by the previous compatibilities. (Note that such a $\delta : W_K \to A^\times$ factors through $W_K^{ab} \to A^\times$, and $W_K^{ab} \xrightarrow{\sim} K^\times$ via art, so we will sometimes think of δ as a character on $K^\times$.)

10.5 Properties of rank 1 modules

Proposition 10.5.1. *Suppose X is a rigid space. Any rank 1 (φ, Γ_K)-module M over $R_X(\pi_K)$ is of the form*

$$\mathscr{L}(\delta) = \mathscr{L} \otimes_{\mathscr{O}_X} R_X(\pi_K)(\delta)$$

where $\delta : K^\times \to R_X(\pi_K)^\times$ is a continuous character and $\mathscr{L} = H^0_{\varphi,\gamma_K}(M(\delta^{-1}))$ is a line bundle. This δ is uniquely determined.

We have

$$D^\circ_{sen}(R_A(\pi_K)(\delta)) = A \otimes K$$

where Θ_{sen} can be written as follows. Our continuous character $\delta : K^\times \to A^\times$ is automatically locally analytic, so we can differentiate it to get $d\delta : K \to A$, which is $\mathbb{Q}_p$-linear. Since tr is nondegenerate on K, we have

$$(d\delta)(\alpha) = \mathrm{tr}_{K/\mathbb{Q}_p}(\alpha \underline{d\delta})$$

for some $\underline{d\delta} \in A \otimes K$. Then Θ_{sen} corresponds to $-\underline{d\delta}$. For example, if $L/\mathbb{Q}_p$ is finite and sufficiently large (containing all embeddings of K into the algebraic closure of L), then we can write

$$d\delta = \sum (d\delta)_\tau \tau : K \to L$$

where $(d\delta)_\tau \in L$, and

$$HTS_\tau(R_L(\pi_K)(\delta)) = \{-(d\delta)_\tau\}.$$

We will sometimes write $(d\delta)_\tau = -wt_\tau(\delta)$.

If $L/\mathbb{Q}_p$ is finite, then $R_A(\pi_K)(\delta)$ is de Rham if and only if $HTS_\tau(R_A(\pi_K)(\delta)) \subset \mathbb{Z}$ (this is special to 1-dimensional representations), which is true if and only if there is an open $U \subset K^\times$ and $m_\tau \in \mathbb{Z}$ for all $\tau : K \hookrightarrow \overline{L}$ such that

$$\delta|_U = \prod_{\tau : K \hookrightarrow \overline{L}} \tau^{-m_\tau}.$$

(This is like our earlier analysis of Galois characters but we are no longer required to send the uniformizer to a p-adic unit.) In this case,

$$R_L(\pi_K)(\delta) = M_{rig}\left(L\left(\delta \prod_\tau \tau^{m_\tau}\right)\right)$$

where by $L(\delta \prod_\tau \tau^{m_\tau})$ we mean L with the action of $\delta \prod_\tau \tau^{m_\tau}$, considered as a WD rep filtered such that $\mathrm{gr}^i_\tau \neq (0)$ if and only if $i = m_\tau$.

11 February 18: cohomology, triangulations, and character varieties.

11.1 Cohomology of rank 1 (φ, Γ)-modules

We were looking at rank 1 (φ, Γ_K)-modules over $R_A(\pi_K)$. We saw that locally on A, these are the same as continuous characters $\delta : K^\times (\cong W_K^{ab}) \to A^\times$; globally they could be twisted by a line bundle on A. We saw that $R_A(\pi_K)(\delta)$ corresponds to a Galois representation if and only if δ has a continuous extension to the Galois group, i.e. $\delta(\varpi_K)$, where ϖ_K is a uniformizer, has all eigenvalues units. In this case $R_A(\pi_K)(\delta) = M_{rig}(A(\hat{\delta}))$ where $\hat{\delta}$ is the unique continuous extension of δ to $G_K^{ab} \supset W_K^{ab}$ (note W_K^{ab} is dense in G_K^{ab}).

δ is automatically locally analytic, and HTS numbers correspond to the derivative of δ at 1.

Proposition 11.1.1. *Let $L/\mathbb{Q}_p$ be a finite extension, sufficiently large to contain all images of $\tau : K \hookrightarrow \overline{L}$. Let $\delta : K^\times \to L^\times$ be a continuous character.*

1. $H^0_{\varphi,\gamma_K}(R_L(\pi_K)(\delta)) = (0)$ *unless* $\delta = \prod_{\tau: K \hookrightarrow \overline{L}} \tau^{-m_\tau}$ *for some* $m_\tau \in \mathbb{Z}_{\geq 0}$*. In this exceptional case,* $H^0_{\varphi,\gamma_K} = L \prod_\tau t_\tau^{m_\tau}$*, where* $t_\tau \in R_L(\pi_K)$ *are certain elements such that* (t_τ) *is well-defined and* $(\prod_\tau t_\tau) = (t)$*. (Note that if* $R_L(\pi_K)(\delta)$ *corresponds to a Galois representation, then we know* $H^0 = 0$ *unless* δ *is trivial, and on the other hand we can only have* $\delta = \prod_{\tau: K \hookrightarrow \overline{L}} \tau^{-m_\tau}$ *if the* m_τ*s are all* 0*, so this checks out.)*

2. $H^2_{\varphi,\gamma_K}(R_L(\pi_K)(\delta)) = (0)$ *unless* $\delta = |\cdot|_K \prod_\tau \tau^{m_\tau}$ *for some* $m_\tau \in \mathbb{Z}_{>0}$*. In this exceptional case, it is isomorphic to* L*.*

3. $\dim_L H^1_{\varphi,\gamma_K}(R_L(\pi_K)(\delta)) = [K : \mathbb{Q}_p] + \dim H^0_{\varphi,\gamma_K} + \dim H^2_{\varphi,\gamma_K}$ *(this is the Euler formula;* $[K : \mathbb{Q}_p]$ *is really multiplied by the rank of the* (φ, Γ)*-module which is* 1*).*

Recall that $H^0_{\varphi,\gamma_K}(M) \cong \mathrm{Hom}(R_L(\pi_K), M)$.

Corollary 11.1.2. $\mathrm{Hom}(R_L(\pi_K)(\delta), R_L(\pi_K)(\delta')) \cong \mathrm{Hom}(R_L(\pi_K), R_L(\pi_K)(\delta'\delta^{-1})) \cong (0)$ *unless* $\delta = \delta' \prod_\tau \tau^{m_\tau}$ *for* $m_\tau \in \mathbb{Z}_{\geq 0}$*. In this case, it equals* L*(multiplication by* $\prod_\tau t_\tau^{m_\tau}$*).*

In particular, (φ, Γ)-modules do not form an abelian category—there can be nontrivial homomorphisms between non-isomorphic rank 1 objects.

Lemma 11.1.3. $\tilde{H}(R_L(\pi_K)(\delta)) = (0)$ *unless* $\delta = \prod \tau^{-m_\tau}$ *for* $m_\tau \in \mathbb{Z}_{\geq 0}$*, in which case it is isomorphic to* L*.*

Proof. We know that $\tilde{H}(R_L(\pi_K)(\delta)) \neq (0)$ if and only if $\delta = \prod \tau^{-m_\tau}$. What we need to prove is that in that case it is at most one-dimensional, i.e.

$$\tilde{H}\left(R_L(\pi_K)\left(\prod_\tau \tau^{-m_\tau}\right)\right) \hookrightarrow L.$$

We have an exact sequence

$$0 \to R_L(\pi_K) \xrightarrow{\text{mult. by } \prod t_\tau^{m_\tau}} R_L(\pi_K)\left(\prod_\tau \tau^{-m_\tau}\right) \to R_L(\pi_K)\left(\prod_\tau \tau^{-m_\tau}\right) / \prod t_\tau^{m_\tau} \to 0.$$

Applying $\tilde{H}$ gives

$$0 \to L \to \tilde{H}\left(R_L(\pi_K)\left(\prod_\tau \tau^{-m_\tau}\right)\right) \to \tilde{H}\left(R_L(\pi_K)\left(\prod_\tau \tau^{-m_\tau}\right)/\prod t_\tau^{m_\tau}\right).$$

So we want to prove that

$$\tilde{H}\left(R_L(\pi_K)\left(\prod_\tau \tau^{-m_\tau}\right)/\prod t_\tau^{m_\tau} R_L(\pi_K)\left(\prod_\tau \tau^{-m_\tau}\right)\right) = (0).$$

But this equals

$$\tilde{H}\left(\prod t_\tau^{-m_\tau} R_L(\pi_K)/R_L(\pi_K)\right) \hookrightarrow \tilde{H}\left(t^{-m} R_L(\pi_K)/R_L(\pi_K)\right)$$

for $m \geq \max(m_\tau)$. For $n \gg 0$, we can Taylor expand this to get an embedding into

$$\bigcup_r \left(\tilde{t}^{-m}(L \otimes K_n)[\![\tilde{t}]\!]/(L \otimes K_n)[\![\tilde{t}]\!]\right)^{\ker(\gamma-1)^r}$$

for γ such that $(\mathrm{Gal}(K_\infty/K_n) =)\Gamma_n = \langle\gamma\rangle$ (we lose the information about being in $\ker(\varphi-1)^r$ in the Taylor expansion, but that's okay; we could choose γ to generate all of Γ_K, but then it would also act on K_n which would be annoying, and it's enough to get that this is already 0 for γ generating a smaller subgroup). We claim this is 0, because an element of

$$\tilde{t}^{-m}(L \otimes K_n)[\![\tilde{t}]\!]$$

can be written as

$$\sum_{a=1}^{m} \beta_\alpha \tilde{t}^{-a} \mapsto^{(\gamma-1)^r} \sum_{a=1}^{m} \beta_a(\epsilon_p(\gamma)^{-a} - 1)^r \tilde{t}^{-a}$$

which is clearly injective since $-a$ is never 0. $\square$

11.2 Triangulations

Definition 11.2.1. Suppose M is a (φ, Γ_K)-module over $R_A(\pi_K)$ of rank n. Let $\underline{\delta} = (\delta_1, \ldots, \delta_n)$ be continuous characters $K^\times \to A^\times$. By a triangulation of M with parameter $\underline{\delta}$, we mean an increasing filtration $\mathrm{Fil}_\bullet$ of M by sub-(φ, Γ_K)-modules (which are in particular projective) such that $\mathrm{Fil}_0 M = (0)$, $\mathrm{Fil}_n M = M$, and $\mathrm{gr}_i M \cong \mathscr{L}_i(\delta_i)$ where $\mathscr{L}_i$ is a locally free A-module of rank 1. We call such an M trianguline with parameter $\underline{\delta}$.

We call such an M strictly trianguline with parameter $\underline{\delta}$ if it is trianguline with triangulation $\mathrm{Fil}_\bullet$ and for all i, $H^0_{\varphi,\gamma_K}((M/\mathrm{Fil}_i M)(\delta_{i+1}^{-1}))$ is locally free of rank 1 over A. (That is, not only is the filtration unique, but at each step of the filtration, there is only one way to take the next step, even allowing the possibility of getting stuck in later steps.)

The following theorem says that triangulations can be sort of globally interpolated, though with various errors.

Theorem 11.2.2. *Suppose X is an integral rigid space, that $\delta_1, \ldots, \delta_n : K^\times \to \mathscr{O}_X(X)^\times$, and that M is a rank n (φ, Γ_K)-module over X. Suppose also $Z \subset X$ is a Zariski dense subset such that for all $z \in Z$, the $R_{k(z)}(\pi_K)$-module M_z is strictly trianguline with parameters $(\delta_{1,z}, \ldots, \delta_{n,z})$. Then there is $f : X' \to X$ proper and birational, with X' again integral (so you "might have to blow up X a bit"), and a filtration $\mathrm{Fil}_\bullet$ on f^*M by (φ, Γ_K)-stable coherent $R_{X'}(\pi_K)$-submodules (which might not be projective, so $\mathrm{Fil}_i f^*M$ may not actually be a (φ, Γ_K)-module), such that*

1. *$Y = \{y \in X' \mid (f^*M)_y$ is not strictly trianguline with parameter $(\delta_{1,y}, \ldots, \delta_{n,y})\}$ is Zariski closed and disjoint from $f^{-1}Z$.*
2. *$\mathrm{gr}_i f^*M$ might not be $\mathscr{L}_i(\delta_i)$, as we would like, but we do have $\mathrm{gr}_i f^*M \hookrightarrow \mathscr{L}_i(\delta_i)$ for some line bundle $\mathscr{L}_i/X'$. Furthermore, the cokernel is supported on Y and locally on X' killed by some power of t (if X' is not quasicompact maybe there is not a power that works everywhere).*

Remark 5. By Zariski dense in a rigid space, we mean that there exists an admissible affinoid covering of the space by $\mathrm{Sp}(A_i)$ such that $Z \cap \mathrm{Sp}(A_i)$ is Zariski dense, in the sense that any $a \in A_i$ vanishing on all of Z is automatically 0. This does not imply that Z is Zariski dense upon intersection with any affinoid. For example, $Z = \{z \in \Delta[a, 1] \mid |z| > 1/p\} \subset \Delta[0, 1]$ is Zariski dense using the admissible affinoid covering $\{\Delta[0, 1]\}$ (in particular, it has infinitely many points), but Z has empty intersection with the affinoid $\Delta[0, 1/p]$. This is different from what we're used to in standard algebraic geometry.

Corollary 11.2.3. *Keep the notation from Theorem 11.2.2. Then for all $x \in X$, there are continuous characters $\delta'_{i,x} : K^\times \to k(x)^\times$ such that*

1. *M_x is trianguline with parameter $(\delta'_{i,x})$.*
2. *$\delta'_{i,x} = \delta_{i,x} \prod_{\tau : K \hookrightarrow \overline{k(x)}} \tau^{m_\tau}$ with $m_\tau \in \mathbb{Z}$.*

Proof. Replace X by X' (if the statement is true for a preimage of $x \in X$ then it's also true for x) and then by $\mathrm{Sp}(A) \subset X'$ containing x such that $\mathscr{L}_i|_{\mathrm{Sp}(A)}$ is free. We are given $\mathrm{Fil}_i M$ and $\mathrm{gr}_i M \hookrightarrow R(\pi_K)(\delta_i)$. The cokernel C_i is killed by some t^N for all i. We have a long exact sequence

$$0 \to \mathrm{Tor}_2^A(C_i, k(x)) \xrightarrow{\sim} \mathrm{Tor}_1^A(\mathrm{gr}_i M, k(x)) \to (0) \to \mathrm{Tor}_1^A(C_i, k(x))$$
$$\to (\mathrm{gr}_i M) \otimes k(x) \to R_{k(x)}(\delta_{i,x}) \to C_i \otimes k(x) \to 0$$

where the (0) in the third term is because $R_A(\pi_K)(\delta_i)$ is flat over A. Here $C_i \otimes k(x)$, $\mathrm{Tor}_1^A(C_i, k(x))$, and $\mathrm{Tor}_2^A(C_i, k(x)) \cong \mathrm{Tor}_1^A(\mathrm{gr}_i M, k(x))$ are killed by t^N, so the kernel and cokernel of

$$(\mathrm{gr}_i M) \otimes k(x) \to R_{k(x)}(\pi_K)(\delta_{i,x})$$

is killed by t^N, so the $R_{k(x)}(\pi_K)$-torsion submodule $((\mathrm{gr}_i M) \otimes k(x))^{tor}$ is killed by t^N. Next look at

$$0 \to \mathrm{Fil}_{i-1} M \to \mathrm{Fil}_i M \to \mathrm{gr}_i M \to 0$$

and again take the associated long exact

$$\mathrm{Tor}_1^A(\mathrm{gr}_i M, k(x)) \to (\mathrm{Fil}_{i-1} M) \otimes k(x) \to (\mathrm{Fil}_i M) \otimes k(x) \to (\mathrm{gr}_i M) \otimes k(x) \to 0$$

in which $\operatorname{Tor}_1^A(\operatorname{gr}_i M, k(x)) \cong \operatorname{Tor}_2^A(C_i, k(x))$ and the torsion inside $(\operatorname{gr}_i M) \otimes k(x)$ are killed by t^N.

Define $\operatorname{Fil}_i(M \otimes k(x))$ to be the saturation of the image of $\operatorname{Fil}_i(M) \otimes k(x) \to M \otimes k(x)$. Suppose we have something in $\operatorname{Fil}_i(M) \otimes k(x)$ and some multiple of it by a power of t lies in $\operatorname{im}((\operatorname{Fil}_{i-1} M) \otimes k(x))$. Then it would be torsion in $(\operatorname{gr}_i M) \otimes k(x)$, hence t^N times it would be 0, hence t^N times it would be in $\operatorname{im}((\operatorname{Fil}_{i-1} M) \otimes k(x))$. That is, for anything in the saturation of $\operatorname{im}((\operatorname{Fil}_{i-1} M) \otimes k(x))$, t^N times it is in $\operatorname{im}((\operatorname{Fil}_{i-1} M) \otimes k(x))$. Continuing by reverse induction on i, we conclude that

$$t^{N(n-i)} \operatorname{Fil}_i(M \otimes k(x)) \subset \operatorname{im}\left((\operatorname{Fil}_i M) \otimes k(x) \to M\right).$$

Similarly, $\ker\left((\operatorname{Fil}_i M) \otimes k(x) \to (\operatorname{Fil}_{i+1} M) \otimes k(x)\right)$ is killed by t^N, and continuing with $i+1, i+2$, etc.,

$$\ker\left((\operatorname{Fil}_i M) \otimes k(x) \to \operatorname{Fil}_i(M \otimes k(x))\right)$$

is killed by $t^{N(n-i)}$. So if $m \in M$, by the first containment $t^{(n-i)N} m$ can be lifted to $(\operatorname{Fil}_i M) \otimes k(x)$, say to $\tilde{m}$, which by the second assertion is unique up to torsion. The image m' of $\tilde{m}$ in $R_{k(x)}(\pi_K)(\delta_i)$ is independent of the choice of $\tilde{m}$ because the latter is torsion-free. Dividing by $t^{(n-i)N}$, we get a map

$$\begin{gathered}\operatorname{Fil}_i(M \otimes k(x)) \to t^{(i-n)N} R_{k(x)}(\pi_K)(\delta_i) \\ m \mapsto t^{(i-n)N} m'.\end{gathered}$$

This induces an embedding

$$\operatorname{gr}_i(M \otimes k(x)) \hookrightarrow t^{(i-n)N} R_{k(x)}(\pi_K)(\delta_i).$$

(Proof: if you have $m \in \operatorname{Fil}_{i-1}(M \otimes k(x))$, then $t^{(n-i+1)N} m$ can be lifted to $t^N \tilde{m} \in (\operatorname{Fil}_{i-1} M) \otimes k(x)$, so the image m' of $\tilde{m}$ in $R_{k(x)}(\pi_K)(\delta_i)$ is 0. On the other hand if $m \in \operatorname{Fil}_i(M \otimes k(x))$ has image 0, then its lifting in $(\operatorname{Fil}_i M) \otimes k(x)$ is in the image of $(\operatorname{Fil}_{i-1} M) \otimes k(x)$, so $t^{(n-i)N} m$ is in the image of $\operatorname{Fil}_{i-1}(M) \otimes k(x)$, so m is in the saturation of the image of $\operatorname{Fil}_{i-1}(M) \otimes k(x)$, so is in $\operatorname{Fil}_{i-1}(M \otimes k(x))$.) We conclude that

$$\operatorname{gr}_i(M \otimes k(x)) \cong R_{k(x)}(\pi_K)\left(\delta_i \prod_\tau \tau^{m_\tau}\right)$$

for some $m_\tau \in \mathbb{Z}_{\leq (n-i)N}$, by the classification of rank 1 (φ, Γ_K)-modules over fields. □

In summary, we start with a family that is strictly trianguline in a Zariski dense set of places, then are able to find a filtration that makes it strictly trianguline almost everywhere, and at the few exceptional places, you can be off by a finite amount measured by a power of t, so that you still end up trianguline but the characters can jump. The theory of the eigencurve is about families of trianguline representations in this sense. So now we want to look at the possible characters δ_i.

11.3 Character varieties

Suppose H is a commutative p-adic Lie group, e.g. $K^\times$, such that there exists a compact open subgroup $U \subset H$ with H/U finitely generated. We discussed the following in [11].

1. if A is affinoid and $\delta : H \to A^\times$ is a continuous character, then δ is locally analytic.

2. there is a smooth, quasi-Stein rigid space $\mathscr{C}(H)$ and a universal continuous character $\delta : H \to \mathscr{O}_{\mathscr{C}(H)}(\mathscr{C}(H))^\times$ such that if X is any rigid space and $\epsilon : H \to \mathscr{O}_X(X)^\times$ is a continuous character, there is a unique map $f : X \to \mathscr{C}(H)$ such that $f^*(\delta) = \epsilon$.

Actually such an H is of the form $H \cong H^{tor} \times \mathbb{Z}_p^d \times \mathbb{Z}^e$, where H^{tor} is finite. For example

$$K^\times = \mu(K) \times \mathbb{Z}_p^{[K:\mathbb{Q}_p]} \times \mathbb{Z}$$

where the last term comes from the uniformizer. We have $\mathscr{C}(H_1 \times H_2) = \mathscr{C}(H_1) \times \mathscr{C}(H_2)$, so we can describe all $\mathscr{C}(H)$s using that

- $\mathscr{C}(\mathbb{Z}) = \mathbb{G}_m^{an} = \varinjlim_{n,m} \operatorname{Sp} \mathbb{Q}_p\langle p^n T, p^m T^{-1}\rangle$ with $\delta(1) = T$,
- $\mathscr{C}(\mathbb{Z}_p) = \Delta[0,1)$ with $\delta(1) = 1 + T$ and $\delta(z) = (1+T)^z = \sum_{i=0}^\infty \binom{z}{i} T^i$, and
- for Δ finite, $\mathscr{C}(\Delta) = \operatorname{Sp} \mathbb{Q}_p[\Delta]$ with $\delta(x) = x$.

We will be especially interested in characters of $(K^\times)^n$. Let $\mathscr{C}_{K,n} = \mathscr{C}((K^\times)^n)$. By restriction, we get a map

$$\mathscr{C}_{K,n} \to \mathscr{C}((\mathcal{O}_K^\times)^n) =: \mathscr{W}_{K,n}$$

(called "weight space"). Also in $\mathscr{C}_{K,n}$ we have the (Zariski open) regular locus $\mathscr{C}_{K,n}^{reg}$ of characters $\delta_1, \ldots, \delta_n$ such that for $i < j$ (note the order), δ_i/δ_j is not of the form $\prod_{\tau : K \hookrightarrow \overline{\mathbb{Q}}_p} \tau^{m_\tau}$ for $m_\tau \in \mathbb{Z}_{\geq 0}$.

Remark 6. If M is trianguline WRT $\underline{\delta} \in \mathscr{C}_{K,n}^{reg}(L)$, then M is strictly trianguline WRT $\underline{\delta}$.

Proof. We will prove by reverse induction on $j > i$ that $H^0_{\varphi,\gamma_K}((M/\operatorname{Fil}_j M)(\delta_{i+1}^{-1})) = (0)$. This is true if $j = n$, because $M/\operatorname{Fil}_n M = (0)$. In general, we have an exact sequence

$$0 \to \operatorname{gr}_{j+1} M \to M/\operatorname{Fil}_j M \to M/\operatorname{Fil}_{j+1} M \to 0$$

where we know $\operatorname{gr}_{j+1} M \cong R_L(\pi_K)(\delta_{j+1})$. The long exact sequence after twisting by δ_{i+1} is

$$0 \to H^0_{\varphi,\gamma_K}(R_L(\pi_K)(\delta_{j+1}/\delta_{i+1})) \to H^0_{\varphi,\gamma_K}((M/\operatorname{Fil}_j M)(\delta_{i+1}^{-1})) \to H^0_{\varphi,\gamma_K}((M/\operatorname{Fil}_{j+1} M)(\delta_{i+1}^{-1})).$$

The first term is (0) if $j > i$ by the regularity assumption, and the last is 0 by the inductive hypothesis, so the middle term is also 0.

Now setting $j = i$, the long exact sequence is

$$0 \to H^0_{\varphi,\gamma_K}(R_L(\pi_K)) \to H^0_{\varphi,\gamma_K}((M/\operatorname{Fil}_i M)(\delta_{i+1}^{-1})) \to (0)$$

where the last term is 0 by the previous claim. We know the first term is L, so we're done. $\square$

Next time, in the case of rank 2 and base field $\mathbb{Q}_p$, we'll explicitly characterize the two-dimensional trianguline representations, and see that the character jumps actually happen in natural families.

12 February 23: 2d triangulines and deformations of triangulines.

Recall that if A is an affinoid $\mathbb{Q}_p$-algebra and $M/R_A(\pi_K)$ is a (φ, Γ_K)-module of rank n, and $\underline{\delta} = (\delta_1, \ldots, \delta_n) : K^\times \to A^\times$ is continuous, we say that M is trianguline with parameter $\underline{\delta}$ if there is a filtration $\operatorname{Fil}_i M$ by sub-(φ, Γ_K)-modules with $\operatorname{Fil}_0 M = (0)$, $\operatorname{Fil}_n M = M$, and $\operatorname{gr}_i M \cong R_A(\pi_K)(\delta_i)$. We say that M is strictly trianguline if given Fil_i there is a unique choice for Fil_{i+1} for all i (expressed as the vanishing of some H^0).

To describe families of such things, we want to describe families of δ_is, so we defined the rigid spaces $\mathscr{C}_{K,n} = \mathscr{C}((K^\times)^n) \to \mathscr{W}_{k,n} = \mathscr{C}((\mathcal{O}_K^\times)^n)$, the second called weight space. We also defined the regular locus $\mathscr{C}_{K,n}^{reg} \subset \mathscr{C}_{K,n}$ of $(\delta_1, \ldots, \delta_n)$ where for $i < j$, δ_i/δ_j is not of the form $\prod_{\tau : K \hookrightarrow \overline{K}} \tau^{m_\tau}$ with $m_\tau \in \mathbb{Z}_{\geq 0}$. We saw that if M is trianguline with a regular parameter $\underline{\delta}$, then M is strictly trianguline.

12.1 2d trianguline representations

Example 12.1.1. Let $K = \mathbb{Q}_p$ and $\operatorname{rank} M = 2$, $p > 2$. What are the possible $(\delta_1, \delta_2) : \mathbb{Q}_p^\times \to A^\times$? We have

$$\mathbb{Q}_p^\times = p^{\mathbb{Z}} \times \mu_{p-1} \times (1 + p\mathbb{Z}_p)$$

so

$$\mathscr{C}(\mathbb{Q}_p^\times \times \mathbb{Q}_p^\times) \cong (\mathbb{G}_m^{an} \times \widehat{\mu_{p-1}} \times \Delta^0)^2$$

where $\Delta^0 = \Delta[0, 1)$ is the open unit disc. A given $\delta : \mathbb{Q}_p^\times \to A^\times$ is determined by $p \mapsto u \in A^\times$, $\psi : \mu_{p-1} \to A^\times$, and $1 + p \mapsto 1 + z$ with $|z| < 1$. We will thus refer to it by (u, ψ, z). We write

$$v(\delta) = \operatorname{val}_p(\delta(p)) = \operatorname{val}_p(u)$$

for the "slope" of δ, and

$$wt(\delta) = -\frac{\log(1+z)}{\log(1+p)}.$$

Consider the locus in $\mathscr{C}(\mathbb{Q}_p^\times \times \mathbb{Q}_p^\times)$ consisting of the following two parts.

A. $v(\delta_1) + v(\delta_2) = 0$, $v(\delta_1) \geq 0$, and M non-split. We have calculated H^1 of the ratio of these things, so we can classify non-split extensions

$$0 \to \delta_1 \to M \to \delta_2 \to 0.$$

over this locus. Usually it is unique, but in some cases the Ext^1 is 2-dimensional, giving a $\mathbb{P}^1$ of possible extensions (since the extension depends only on the choice of line). We are going to blow up at those points (this can be made rigorous for rigid spaces but we won't), that is, blow up with exceptional divisor $\mathbb{P}^1$ when $\delta_1 = \delta_2$ or $\delta_1 = |\cdot|_p \tau^k \delta_2$ for $k \in \mathbb{Z}_{\geq 1}$, where $\tau : \mathbb{Q}_p \hookrightarrow A$ is the natural map.

B. the split locus where $v(\delta_1) = v(\delta_2) = 0$ and $M = \delta_1 \oplus \delta_2$.

A specializes to B in some "funny stacky way", but we will just treat them separately for simplicity.

This locus includes all M such that M is Galois, i.e. equal to $M_{rig}(V)$ for some Galois representation V. (There is a criterion for when something can be Galois using slopes, which we have not described, and which approximately translates to the above conditions.) In fact it is the Zariski closure of those M such that M is Galois. This is the locus that interests us, since we're really interested in Galois representations, not all (φ, Γ)-modules.

M is a twist of $M_{rig}(D)$ for a filtered WD rep D if and only if in case A, $wt(\delta_2) - wt(\delta_1) \in \mathbb{Z}_{>0}$, or in case B, $wt(\delta_2) - wt(\delta_1) \in \mathbb{Z}$. (This is because the difference in weights comes from the jump in the filtration of D, which has integer steps.)

When is D admissible? TFAE:

1. M is a twist (tensored by a 1-dimensional object—the interesting phenomena here don't really depend on twists by characters, so it's more natural just to work up to twists) of $M_{rig}(D)$ with D an admissible filtered WD rep.

2. M is a twist of $M_{rig}(V)$ for V a de Rham Galois representation.

3. M is a twist both of $M_{rig}(D)$ and $M_{rig}(V)$ for V a Galois representation and D a filtered WD rep. (We already saw the equivalence of these first three things before.)

4. In case A, the difference $wt(\delta_2) - wt(\delta_1)$, which we know must be ≥ 0, is in fact $\geq v(\delta_1) \geq 0$ (this comes out of the admissibility condition). In case B, $wt(\delta_2) - wt(\delta_1) \in \mathbb{Z}$.

These are the de Rham representations sitting inside the ones coming from filtered modules. What are the possible filtrations? The image of M_{rig} of filtered WD reps is closed under subobjects, so in this case, filtrations (triangulations) on M are the same as filtrations on D (respecting the WD rep). If the WD rep is irreducible, there are none, but then M can't be trianguline, so that doesn't happen. Otherwise the WD rep is the sum of two characters, or it has nontrivial N. So there are

- 2 filtrations in the potentially crystalline case (coming from the choice of one of the characters), and

- 1 in the potentially semistable but not potentially crystalline case (the only possible sub is $\ker(N)$), unless

- it is potentially crystalline with equal characters, in which case there are ∞ or 1 depending on how you count (you can choose the submodule however you want, but they're all the same).

Which M are Galois? TFAE:

1. M is not a twist of Galois (i.e. twist of $M_{rig}(V)$ for V a Galois rep).

2. $M = M_{rig}(D)$ with D not admissible. (So you can cover the whole space by either things that are Galois or things that come from filtered WD reps! This is probably a coincidence, though.)

3. $v(\delta_1) > wt(\delta_2) - wt(\delta_1) = k \in \mathbb{Z}_{>0}$ in case A. (This doesn't happen in case B.)

If you put a family of Galois representations over this space, what is the specialization of the (φ, Γ)-module at these bad points? It can't specialize to the (φ, Γ)-module we were "expecting" from the Galois representation. This is consistent with what we said about how if you have a family of trianguline (φ, Γ)-modules, but on the exceptional locus it can be trianguline with different parameters from the ones you expect. In particular, the specialization of the (φ, Γ)-module should be

$$0 \to \delta_1 \tau^{-k} \to M' \to \delta_2 \tau^k \to 0$$

for $\tau : \mathbb{Q}_p \hookrightarrow L$. So the phenomenon of jumping parameters we discussed last time occurs naturally. This is the same (φ, Γ)-module you would find at $(\delta_1 \tau^{-k}, \delta_2 \tau^k, M')$.

So our family $\mathscr{T}$ of trianguline (φ, Γ)-modules maps to Galois representations, taking the corresponding Galois representation at most points. The fibers are singletons except in the following cases.

1. We saw that in the potentially crystalline case, there are two filtrations which will correspond to two things giving the same Galois representation. This happens in case A, where if $k = wt(\delta_2) - wt(\delta_1) \geq v(\delta_1) \geq 0 \in \mathbb{Z}_{>0}$ and M is potentially crystalline, the triangulations
$$(\delta_1, \delta_2, 0 \to \delta_1 \to M \to \delta_2 \to 0)$$
$$(\delta_2 \tau^k, \delta_1 \tau^{-k}, M)$$
both map to M (if one of these is in case A then the other is too).

2. The non-Galois case we already discussed. This happens in case A, where if $v(\delta_1) > k = wt(\delta_2) - wt(\delta_1) \in \mathbb{Z}_{>0}$, the triangulations
$$(\delta_1, \delta_2, 0 \to \delta_1 \tau^{-k} \to M \to \delta_2 \tau^k \to 0)$$
$$(\delta_1 \tau^{-k}, \delta_2 \tau^k, M)$$
pair with the ones that are

3. still in case A, but have $wt(\delta_2) - wt(\delta_1) = k \in \mathbb{Z}_{<0}$ and look like
$$(\delta_1, \delta_2, 0 \to \delta_1 \to M \to \delta_2 \to 0)$$
$$(\delta_1 \tau^{-k}, \delta_2 \tau^k, M).$$

4. In case B, if $\delta_1 \neq \delta_2$, clearly $(\delta_1, \delta_2, \delta_1 \oplus \delta_2)$ and $(\delta_2, \delta_1, \delta_1 \oplus \delta_2)$ have the same image.

(Richard hasn't seen anything explicit like this worked out in greater generality, though with more work it should be possible.)

12.2 Deformations of trianguline representations

Let $L/\mathbb{Q}_p$ be a finite extension, and Art_L the category of artinian local L-algebras with residue field L. (In particular they are affinoid algebras.) Let $r : G_K \to GL_n(L)$ be such that $M_{rig}(r)$ is trianguline with parameters $\delta_1, \dots, \delta_n$ (and triangulation Fil_i) for $\delta_1, \dots, \delta_n : K^\times \to L^\times$ continuous. (We will just say that r is trianguline with parameters $\delta_1, \dots, \delta_n$.)

Definition 12.2.1. By a lifting $\tilde{r}$ of r to A we mean a continuous $\tilde{r} : G_K \to GL_n(A)$ with $\tilde{r} \pmod{\mathfrak{m}_A} = r$. We say that $\tilde{r} \sim \tilde{r}'$ are equivalent if there is $g \in \mathrm{id}_n + M_{n\times n}(\mathfrak{m}_A)$ with $\tilde{r}' = g\tilde{r}g^{-1}$.

By a lifting $(\tilde{r}, \underline{\tilde{\delta}}, \widetilde{\mathrm{Fil}}_i)$ of $(r, \underline{\delta}, \mathrm{Fil}_i)$ to A we mean

1. a lifting $\tilde{r}$ of r,
2. liftings $\tilde{\delta}_i : K^\times \to A^\times$ of $\delta_i : K^\times \to L^\times$ for all i, and
3. a triangulation $\widetilde{\mathrm{Fil}}_i$ of $M_{rig}(\tilde{r})$ lifting Fil_i

with equivalence $\sim$ defined in the same way (there is a matrix g that takes $\tilde{r}$ to $\tilde{r}'$ and also each $\widetilde{\mathrm{Fil}}_i$ to $\widetilde{\mathrm{Fil}}'_i$).

Lemma 12.2.2. *If $\underline{\delta} \in \mathscr{C}^{reg}_{K,n}$ is regular, and $(\tilde{r}, \tilde{\delta}_i, \widetilde{\mathrm{Fil}}_i)$ and $(\tilde{r}', \tilde{\delta}'_i, \widetilde{\mathrm{Fil}}'_i)$ lift $(r, \delta_i, \mathrm{Fil}_i)$, then $\tilde{\delta}_i = \tilde{\delta}'_i$ and $\widetilde{\mathrm{Fil}}_i = \widetilde{\mathrm{Fil}}'_i$.*

Proof. Because $\underline{\delta}$ is regular, we have $\mathrm{Hom}(R_L(\pi_K)(\delta_i), R_L(\pi_K)(\delta_j)) = (0)$ if $j > i$.

Now we make a devissage argument to show that $\mathrm{Hom}(R_A(\pi_K)(\tilde{\delta}_i), R_A(\pi_K)(\tilde{\delta}'_j)) = (0)$ for $j > i$. Specifically, we induct on the length of A. Choose $(0) \subsetneq I \subsetneq A$ where the length of I is 1. Look at

$$0 \to R_A(\pi_K)(\tilde{\delta}_j) \otimes_A I \to R_A(\pi_K)(\tilde{\delta}'_j) \to R_{A/I}(\pi_K)(\tilde{\delta}'_j) \to 0.$$

We have $R_A(\pi_K)(\tilde{\delta}_j) \otimes_A I \cong R_L(\pi_K)(\delta_j)$, giving the left exact

$$\begin{aligned} 0 \to \mathrm{Hom}(R_A(\pi_K)(\tilde{\delta}_i), R_L(\pi_K)(\delta_j)) &\to \mathrm{Hom}(R_A(\pi_K)(\tilde{\delta}_i), R_A(\pi_K)(\tilde{\delta}_j)) \\ &\to \mathrm{Hom}(R_A(\pi_K)(\tilde{\delta}_i), R_{A/I}(\pi_K)(\tilde{\delta}'_j)). \end{aligned}$$

To show that the middle term is 0, it suffices to prove that the exterior terms are 0. Continuing with $I \supset I' \supset A$ we see that it suffices to prove that

$$\mathrm{Hom}(R_A(\pi_K)(\tilde{\delta}_i), R_L(\pi_K)(\delta_j)) = 0.$$

Now repeat: we have

$$0 \to R_L(\pi_K)(\delta_i) \to R_A(\pi_K)(\tilde{\delta}_i) \to R_{A/I}(\pi_K)(\tilde{\delta}_i) \to 0$$

giving the long exact

$$\begin{aligned} 0 \to \mathrm{Hom}(R_{A/I}(\pi_K)(\tilde{\delta}_i), R_L(\pi_K)(\delta_j)) &\to \mathrm{Hom}(R_A(\pi_K)(\tilde{\delta}_i), R_L(\pi_K)(\delta_j)) \\ &\to \mathrm{Hom}(R_L(\pi_K)(\delta_i), R_L(\pi_K)(\delta_j)) \end{aligned}$$

and continuing with $I \subsetneq I' \subsetneq A$, we see it suffices to prove that

$$\mathrm{Hom}(R_L(\pi_K)(\delta_i), R_L(\pi_K)(\delta_j)) = (0)$$

which we already know. By a similar devissage,

$$\mathrm{Hom}_A(R_A(\pi_K)(\tilde{\delta}_i), M_{rig}(\tilde{r})/\widetilde{\mathrm{Fil}}'_{i-1}M_{rig}(\tilde{r})) = \mathrm{Hom}_A(R_A(\pi_K)(\tilde{\delta}_i), \tilde{\mathrm{gr}}'_i M_{rig}(\tilde{r})).$$

This is because $\tilde{\mathrm{gr}}'_i M_{rig}(\tilde{r})$ is a sub in $M_{rig}(\tilde{r})/\widetilde{\mathrm{Fil}}'_{i-1}M_{rig}(\tilde{r})$, so writing out the resulting exact sequence, we conclude it suffices to prove that

$$\mathrm{Hom}_A(R_A(\pi_K)(\tilde{\delta}_i), M_{rig}(\tilde{r})/\widetilde{\mathrm{Fil}}'_i M_{rig}(\tilde{r})) = (0)$$

and then in fact that

$$\mathrm{Hom}(R_A(\pi_K)(\tilde{\delta}_i), \tilde{\mathrm{gr}}'_j M_{rig}(\tilde{r})) = (0)$$

for $j > i$, but $\tilde{\mathrm{gr}}'_j M_{rig}(\tilde{r}) = R_A(\pi_K)(\tilde{\delta}'_j)$ so this is true. (The principle is that there is no interaction between things with δ_i and things with δ_j for $j > i$—the homomorphisms over L vanish and then any deformation also vanishes, and this is what people call devissage.)

Now argue by induction on i that $\tilde{\delta}_i = \tilde{\delta}'_i$ and $\widetilde{\mathrm{Fil}}_i = \widetilde{\mathrm{Fil}}'_i$. Suppose it is true for $i-1$. We have

$$\mathrm{Hom}(R_A(\pi_K)(\tilde{\delta}_i), M_{rig}(\tilde{r})/\widetilde{\mathrm{Fil}}_{i-1}M_{rig}(\tilde{r}))$$

where we know $\widetilde{\mathrm{Fil}}_{i-1} = \widetilde{\mathrm{Fil}}'_{i-1}$ by the inductive hypothesis. We know from previously that this equals

$$\mathrm{Hom}(R_A(\pi_K)(\tilde{\delta}_i), R_A(\pi_K)(\tilde{\delta}_i)) = A$$

but also that it equals

$$\mathrm{Hom}(R_A(\pi_K)(\tilde{\delta}_i), \tilde{\mathrm{gr}}'_i M_{rig}(\tilde{r}))$$

from which we conclude that $\tilde{\mathrm{gr}}_i M_{rig}(\tilde{r}) \subset \tilde{\mathrm{gr}}'_i M_{rig}(\tilde{r})$ in $M_{rig}(\tilde{r})/\,\widetilde{\mathrm{Fil}}_{i-1}\, M_{rig}(\tilde{r})$. By symmetry,

$$R_A(\pi_K)(\tilde{\delta}_i) = \tilde{\mathrm{gr}}_i M_{rig}(\tilde{r}) = \tilde{\mathrm{gr}}'_i M_{rig}(\tilde{r}) = R_A(\pi_K)(\tilde{\delta}'_i)$$

so $\tilde{\delta}_i = \tilde{\delta}'_i$, and $\widetilde{\mathrm{Fil}}_i = \widetilde{\mathrm{Fil}}'_i$. $\square$

Continue to assume that $\underline{\delta} \in \mathscr{C}^{reg}_{K,n}$ and that r is trianguline with parameter $\underline{\delta}$. Let $\tilde{r}$ be a lift of r to A. Let

$$\mathscr{I} = \{I \subset A \mid (A/I, \tilde{r} \pmod{I}) \text{ is trianguline lifting } (r, \underline{\delta})\}.$$

1. If $I_1, I_2 \in \mathscr{I}$, then $I_1 \cap I_2 \in \mathscr{I}$. Proof: by the Chinese Remainder Theorem,

$$A/I_1 \cap I_2 = A/I_1 \times_{A/(I_1+I_2)} A/I_2.$$

By our previous lemma, $\widetilde{\mathrm{Fil}}^1_\bullet \pmod{I_1 + I_2} = \widetilde{\mathrm{Fil}}^2_\bullet \pmod{I_1 + I_2}$. So we can define a filtration

$$\widetilde{\mathrm{Fil}}_\bullet = \widetilde{\mathrm{Fil}}^1_\bullet \times_{A/(I_1+I_2)} \widetilde{\mathrm{Fil}}^2_\bullet$$
$$\tilde{\delta}_i = \tilde{\delta}^1_i \times \tilde{\delta}^2_i$$

since $\tilde{\delta}^2_i \pmod{I_1 + I_2} = \tilde{\delta}^1_i \pmod{I_1 + I_2}$.

2. If $\{I_i\}$ form a nested set of ideals in $\mathscr{I}$ then $\bigcap I_i \in \mathscr{I}$. (This is easy—just use uniqueness.)

3. By Zorn's lemma, $\mathscr{I}$ has a minimal element I_0. Furthermore, it is globally minimal: if $I \in \mathscr{I}$ then $I \cap I_0 \in \mathscr{I}$, so $I \cap I_0 = I_0$, so $I \supset I_0$. That is, there is $I_0 \in \mathscr{I}$ such that $\mathscr{I} = \{I \subset A \mid I \supset I_0\}$.

To make trianguline representations behave nicely, there's one more thing we want.

Lemma 12.2.3. *Suppose $A \hookrightarrow B$ in Art_L and $M/R_A(\pi_K)$ is a (φ, Γ_K)-module. Suppose $M \otimes_A B$ is trianguline with parameter $\underline{\tilde{\delta}}$ and $\underline{\tilde{\delta}} \pmod{\mathfrak{m}_B}$ is regular. Then $\tilde{\delta}_i$ is valued in $A^\times$ and M is trianguline with parameter $\underline{\tilde{\delta}}$.*

We'll prove this next time, but here's a consequence: suppose $r : G_K \to GL_n(L')$ is trianguline with regular parameter $\underline{\delta}$. Suppose also that $\tilde{r}$ lifts r to $A \in Art_L$. Then there is a unique ideal $I_0 \subset A$ such that if $f : A \to B$ in Art_L, then $f(\tilde{r})$ is trianguline with parameter lifting $\underline{\delta}$ if and only if $\ker f \supset I_0$. (This is saying that in the regular case, the deformation problem for trianguline representations is a closed sub-problem of all liftings of r.)

12.3 Tangent space and general comments

Consider the space $D_r(L[\epsilon]/(\epsilon^2))$ of deformations (equivalence classes of liftings) of r to $L[\epsilon]/(\epsilon^2)$. If there were a universal deformation, this would measure the tangent space to that. It is a well-known calculation that this is $H^1_{cts}(G_K, \operatorname{ad} r)$, with $[\varphi] \in H^1_{cts}(G_K, \operatorname{ad} r)$ corresponding to $(1 + \epsilon\varphi)r$.

Inside this space we have the (closed) subset (if r is trianguline with regular parameter $\underline{\delta}$) $D_{r,\underline{\delta},\mathrm{Fil}_\bullet}(L[\epsilon]/(\epsilon^2))$, which corresponds inside $H^1_{cts}(G_K, \operatorname{ad} r)$ to the subset $H^1_{tr,\underline{\delta}}(G_K, \operatorname{ad} r)$. Actually this is a subspace. Proof: if $(1+\epsilon\varphi)r$ and $(1+\epsilon\varphi')r$ are trianguline, then $(1+\epsilon_1\varphi+\epsilon_2\varphi')r$ is trianguline over $L[\epsilon_1, \epsilon_2]/(\deg 2) = L[\epsilon_1]/(\epsilon_1^2) \times_L L[\epsilon_2]/(\epsilon_2^2)$. Take the map of rings from $L[\epsilon_1, \epsilon_2]/(\deg 2)$ to $L[\epsilon]/(\epsilon^2)$ given by $\epsilon_1 \mapsto \epsilon$, $\epsilon_2 \mapsto \lambda\epsilon$; then we see that $(1+\epsilon(\varphi+\lambda\varphi'))r$ is trianguline.

The moduli space of trianguline representations lives over all of weight space, so the HT numbers have no particular reason to be rational integers, and you can have large continuous families of them. This is not the case for e.g. de Rham representations, for which a p-adic family of them would have to have fixed HT numbers.

When are Galois representations trianguline? If I have a (φ, Γ)-module that comes from some WD rep, then submodules of the (φ, Γ)-module are in 1-1 correspondence with submodules of the WD rep, so you can completely see the triangulation on that side. Trianguline means filtered by 1-dimensional (φ, Γ)-modules, so you have to be able to filter the WD rep by 1-dimensional things, so it becomes crystalline over an abelian extension. So if you have a 2-dimensional irreducible piece of the WD rep, then there's no hope of it being trianguline. For example, if you have a modular form that's supercuspidal at p, then the corresponding Galois representation wouldn't be trianguline.

What does δ mean in the case of GL_2? After fixing the central character or otherwise removing one variable by twisting, you just get one character δ that's important. This δ

comes from the weight, the nebentypus, and the eigenvalue of U_p. That is, you can read off the δ from the information of the modular form. Note that if you have a form of level prime to p, then go to $\Gamma_0(p)$, there are two possible U_p-eigenvalues corresponding to the two roots of the Hecke polynomial. This exactly corresponds to the fact that the same Galois representation has two triangulations. So you see both of them in the trianguline space, and the Hecke operators away from p are the same but the Hecke operator at p is different. If you pick a particular U_p-eigenvalue that's like fixing a triangulation.

13 February 25: more trianguline deformations.

13.1 Recap

Last time, we had a finite extension $L/\mathbb{Q}_p$, along with the finite extension $K/\mathbb{Q}_p$ as usual, and a representation $r : G_K \to GL_n(L)$ which we assumed was trianguline of regular parameter $\underline{\delta} \in \mathscr{C}^{reg}_{K,n}(L)$, so $\underline{\delta}$ consists of continuous characters $(\delta_1, \dots, \delta_n) : K^\times \to L^\times$ such that if $i < j$ then $\delta_i/\delta_j \neq \prod_{\tau : K \hookrightarrow \overline{L}} \tau^{m_\tau}$ for $m_\tau \in \mathbb{Z}_{\geq 0}$. We saw that there exists a unique filtration Fil_i of $M_{rig}(r)$ such that $\mathrm{gr}_i\, M_{rig}(r) \cong R_L(\pi_K)(\delta_i)$.

Next we considered an artinian algebra $A \in Art_L$ and wanted to know whether a lift $\tilde{r} : G_K \to GL_n(A)$ of r would be trianguline. We showed that there was a unique ideal $I_0 \subset A$ such that $\tilde{r} \pmod{I_0}$ is trianguline with parameters $\tilde{\delta}_1, \dots, \tilde{\delta}_n$ where $\tilde{\delta}_i$ lifts δ_i (note we're not fixing $\tilde{\delta}_i$ at the beginning), and such that if $\tilde{r} \pmod{I}$ is trianguline with parameters lifting δ_i then $I \supset I_0$. In this case $\tilde{\delta}_1, \dots, \tilde{\delta}_n$ and $\widetilde{\mathrm{Fil}}_i$ are uniquely determined. (That is, "if you can deform the triangulation then it deforms uniquely, and the triangulation deforming is a closed condition on the space of Galois representations".)

The following statement about changing the algebra A was a corollary of a lemma we stated but didn't prove.

Corollary 13.1.1. *If $f : A \hookrightarrow B$ in Art_L and $\tilde{r} : G_K \to GL_n(A)$ lifts $r : G_K \to GL_n(L)$ which is trianguline with parameter $\underline{\delta} \in \mathscr{C}^{reg}_{K,n}(L)$, and if $\tilde{r} \otimes_A B$ is trianguline with parameter lifting $\underline{\delta}$, then $\tilde{r}$ is already trianguline with parameter lifting $\underline{\delta}$. That is, $f(\tilde{r})$ is trianguline if and only if* $\ker f \supset I_0$.

This followed from the following lemma.

Lemma 13.1.2. *Let $f : A \hookrightarrow B$ in Art_L and M be a (φ, Γ_K)-module over $R_A(\pi_K)$. Let* Fil_i *be a triangulation of $M \otimes_A B$ with parameter $(\tilde{\delta}_i)$ for $\tilde{\delta}_i : K^\times \to B^\times$ and suppose ($\delta_i = \tilde{\delta}_i$* $\pmod{\mathfrak{m}_B}$) *is regular.*

Then $\tilde{\delta}_1$ is valued in $A^\times$ and there is $R_A(\pi_K)(\tilde{\delta}_i) \hookrightarrow M$ with saturated image over $R_L(\pi_K)$ such that $R_A(\pi_K)(\tilde{\delta}_1) \otimes_A B \xrightarrow{\sim}$ $\mathrm{Fil}_1(M \otimes_A B)$.

That is, the first step in the filtration is already defined over A, and then for the corollary you just mod out by that and keep going.

13.2 Proof of the lemma

Proof of Lemma 13.1.2. WLOG $\delta_1 = 1$, because we can lift it to a map to $A^\times$ somehow and tensor everything by the inverse of that lifting, giving a new set of characters with the same properties.

First we claim that if X is a finite-length A-module then

$$\dim_L \tilde{H}(M \otimes_A X) \leq (\dim_L X)[K_{\infty,0} : \mathbb{Q}_p].$$

Proof by devissage. Filter X by A-submodules such that each graded piece is isomorphic to L as an A-module. Since M is free over A, given $X' \subset X$, the exact sequence

$$0 \to X' \to X \to X/X' \to 0$$

gives another exact sequence

$$0 \to M \otimes_A X' \to M \otimes_A X \to M \otimes_A X/X' \to 0$$

and hence, applying $\tilde{H}$,

$$0 \to \tilde{H}(M \otimes_A X') \to \tilde{H}(M \otimes_A X) \to \tilde{H}(M \otimes_A X/X').$$

Argue by induction on the length of X: if we know the first and third terms satisfy the given bound then the second does too. So WLOG $X = L$.

Now note that $M \otimes_A B$ is free over $R_B(\pi_K)$ with basis e_i such that $\mathrm{Fil}_i = \langle e_1, \ldots, e_i \rangle$; consequently mod $\mathfrak{m}_B$, $M/\mathfrak{m}_A M = (M \otimes_A B)/\mathfrak{m}_B(M \otimes_A B)$ has a filtration $\overline{\mathrm{Fil}}_i = \langle e_1, \ldots, e_i \rangle$ such that $\overline{\mathrm{gr}}_i M/\mathfrak{m}_A M \cong R_L(\pi_K)(\delta_i)$. That is, $M/\mathfrak{m}_A M$ is trianguline with parameter (δ_i), because it's something trianguline reduced $\pmod{\mathfrak{m}_B}$.

Now we devissage again. We have $M \otimes_A L = M/\mathfrak{m}_A M$, and we use the filtration on $M/\mathfrak{m}_A M$. We know that

$$\tilde{H}(R_L(\pi_K)(\delta_i)) = \begin{cases} 0 & i > 1 \\ L \otimes K_{\infty,0} & i = 1 \end{cases}$$

by regularity, so we're done.

We next claim that we have an embedding

$$\begin{gathered} B \otimes_{\mathbb{Q}_p} K_{\infty,0} \hookrightarrow \tilde{H}(R_B(\pi_K)(\tilde{\delta}_1)) \\ 1 \mapsto 1. \end{gathered}$$

That is, we need to check that $1 \in R_B(\pi_K)(\tilde{\delta}_1)$ is killed by $(\varphi - 1)^N$ and $(\gamma - 1)^N$ for some N. But we know that $(\varphi - 1)$ and $(\gamma - 1)$ kill $1 \in R_L(\pi_K)(\delta_1 = 1)$, so if we filter B by $\mathfrak{m}_B^j$, we find that $(\varphi - 1)^j$ and $(\gamma - 1)^j$ will take 1 to $\mathfrak{m}_B^j R_B(\pi_K)(\tilde{\delta}_1)$. So setting N to the length of B gives the claim.

Now we have $\tilde{H}(R_B(\pi_K)(\tilde{\delta}_1)) \subset \tilde{H}(M \otimes_A B)$. But then

$$B \otimes_{\mathbb{Q}_p} K_{\infty,0} \xrightarrow{\sim} \tilde{H}(M \otimes_A B)$$

(because we know that $\dim_L \tilde{H}(M \otimes_A B) \le (\dim_L B)[K_{\infty,0} : \mathbb{Q}_p]$ but $\dim_L B \otimes_{\mathbb{Q}_p} K_{\infty,0} = (\dim_L B)[K_{\infty,0} : \mathbb{Q}_p]$). Therefore

$$\tilde{H}(M \otimes_A B) \otimes_{B \otimes_{\mathbb{Q}_p} K_{\infty,0}} R_B(\pi_K) \xrightarrow{\sim} \mathrm{Fil}_1(M \otimes_A B)$$

(because $\tilde{H}(R_B(\pi_K)(\tilde{\delta}_1))$ is the first filtered piece. This is why we introduced $\tilde{H}$—because in a situation like this it picks out the smallest filtered piece.) So now on B we have a way of describing the first filtered piece without knowing what it was ahead of time, and now we're going to hope that $\tilde{H}$ has the same effect on M originally.

Now consider $0 \to A \to B \to B/A \to 0$. Take $\otimes_A M$ to get

$$0 \to \tilde{H}(M) \to \tilde{H}(M \otimes_A B) \to \tilde{H}(M \otimes_A (B/A))$$

where we know the middle term is $B \otimes_{\mathbb{Q}_p} K_{\infty,0}$ and the last term has $\dim_L$ bounded above by $(\dim_L B/A)[K_{\infty,0} : \mathbb{Q}_p]$. Subtracting gives

$$\dim_L \tilde{H}(M) \ge \dim_L A[K_{\infty,0} : \mathbb{Q}_p]$$

but we already know the inequality in the other direction, so actually this is an equality (and the last map is a surjection but we don't need that). Next from

$$0 \to \mathfrak{m}_A M \to M \to M/\mathfrak{m}_A M \to 0,$$

we get

$$0 \to \tilde{H}(\mathfrak{m}_A M) \to \tilde{H}(M) \to \tilde{H}(M/\mathfrak{m}_A M)$$

where now we know $\dim_L \tilde{H}(M) = \dim_L A[K_{\infty,0} : \mathbb{Q}_p]$, $\tilde{H}(M/\mathfrak{m}_A M) = L \otimes_{\mathbb{Q}_p} K_{\infty,0}$, and $\dim_L \tilde{H}(\mathfrak{m}_A M) \le \dim_L \mathfrak{m}_A[K_{\infty,0} : \mathbb{Q}_p]$. From this we see again that the inequality is an equality and that $\tilde{H}(M) \twoheadrightarrow \tilde{H}(M/\mathfrak{m}_A M)$.

Choose $v \in \tilde{H}(M)$ that goes to a generator 1 of $\tilde{H}(M/\mathfrak{m}_A M)$ over $L \otimes K_{\infty,0}$. We claim that v generates $\tilde{H}(M)$.

We know M is free over $A \otimes K_{\infty,0}$ because it's free over $R_A(\pi_K)$ which is free over $A \otimes K_{\infty,0}$. Say it has basis $\{e_i\}$. Write $v = \sum \lambda_i e_i$. Write $L \otimes K_{\infty,0} = \prod L_j$. Then for all j there is $i(j)$ such that $\lambda_{i(j)} \mapsto$ (a nonzero element of L_j), i.e. $\lambda_{i(j)} \in (A \otimes_L L_j)^\times$. Therefore

$$\begin{aligned} A \otimes_{\mathbb{Q}_p} K_{\infty,0} &\to \tilde{H}(M) \\ a &\mapsto av \end{aligned}$$

is an injection, because $A \otimes_{\mathbb{Q}_p} K_{\infty,0} = \prod_j A \otimes_L L_j$, so we can check it's an injection after projection onto each L_j component, and there it's an injection because the coefficient of $e_{i(j)}$ is a unit in $A^\times$. But both sides have dimension $\dim_L A[K_{\infty,0} : \mathbb{Q}_p]$ so it is an isomorphism.

Okay, now look at $\tilde{H}(M) \otimes_A B \to \tilde{H}(M \otimes B)$. This is an isomorphism $\pmod{\mathfrak{m}_B}$, hence surjective by Nakayama, hence an isomorphism because the $\dim_L$ of both sides is equal. So then in

$$\tilde{H}(M) \otimes_{A \otimes K_{\infty,0}} R_A(\pi_K) \hookrightarrow \tilde{H}(M) \otimes_{A \otimes K_{\infty,0}} R_B(\pi_K)$$

by that isomorphism, the target is

$$= \tilde{H}(M \otimes B) \otimes_{B \otimes K_{\infty,0}} R_B(\pi_K) = \mathrm{Fil}_1(M \otimes B) \subset M \otimes B$$

but the resulting map $\tilde{H}(M) \otimes_{A\otimes K_{\infty,0}} R_A(\pi_K) \hookrightarrow M \otimes B$ also factors as

$$\tilde{H}(M) \otimes_{A\otimes K_{\infty,0}} R_A(\pi_K) \to M \hookrightarrow M \otimes B$$

so $\tilde{H}(M) \otimes_{A\otimes K_{\infty,0}} R_A(\pi_K) \to M$ must be an injection. The source is free of rank 1 over $R_A(\pi_K)$, so isomorphic to $R_A(\pi_K)(\delta_1')$ for some $\delta_1' : K^\times \to A^\times$ such that

$$R_B(\pi_K)(\delta_1') = R_A(\pi_K)(\delta_1') \otimes_A B = R_B(\pi_K)(\tilde{\delta}_1).$$

Therefore $\delta_1' = \tilde{\delta}_1$ is valued in $A^\times$, and we have found $R_A(\pi_K)(\delta_1') \subset M$ such that

$$R_A(\pi_K)(\delta_1') \otimes_A B \xrightarrow{\sim} \mathrm{Fil}_1(M \otimes B)$$

which is what we wanted. □

13.3 De Rham deformations

Let $r : G_K \to GL_n(L)$ be trianguline with regular parameter $\underline{\delta}$, i.e. $M_{rig}(r)$ is trianguline with parameter $\underline{\delta}$. Recall that we were looking at the deformations $Def_r(L[\epsilon]/(\epsilon^2))$, which are in correspondence with $H^1_{cts}(G_K, \mathrm{ad}\, r)$. Also $Def_r(L[\epsilon]/(\epsilon^2))$ contains $Def_{r,\underline{\delta},tri}(L[\epsilon]/(\epsilon^2))$, which are in correspondence with $H^1_{tr,\underline{\delta}}(G_K, \mathrm{ad}\, r)$, which is a subspace of $H^1_{cts}(G_K, \mathrm{ad}\, r)$. We have a natural map

$$Def_{r,\underline{\delta},tri}(L[\epsilon]/(\epsilon^2)) \to Def_{\underline{\delta}}(L[\epsilon]/(\epsilon^2)) \to T_{\underline{\delta}}\mathscr{C}_{K,n} \to T_{\underline{\delta}}\mathscr{W}_{K,n},$$

giving a map $H^1_{tr,\underline{\delta}}(G_K, \mathrm{ad}\, r) \to T_{\underline{\delta}}\mathscr{W}_{K,n}$.

Lemma 13.3.1. *Suppose in addition that r is de Rham and that for all $\tau : K \hookrightarrow \overline{L}$, $wt_\tau(\delta_1) < \cdots < wt_\tau(\delta_n)$. (Note that this assumption implies that $\underline{\delta}$ is regular. UPDATE: sorry, this is false. We should assume in addition that $\underline{\delta}$ is regular.) This condition is sometimes called δ being "non-critical". Then*

$$\ker(H^1_{tr,\underline{\delta}}(G_K, \mathrm{ad}\, r) \to T_{\underline{\delta}}\mathscr{W}_{K,n}) \subset H^1_g(G_K, \mathrm{ad}\, r) := \ker(H^1(G_K, \mathrm{ad}\, r) \to H^1(G_K, \mathrm{ad}\, r \otimes B_{dR})).$$

That is, $H^1_g(G_K, \mathrm{ad}\, r)$ parameterizes the (infinitesimal) de Rham deformations of r.

The point is that if r is de Rham and noncritical and we make an infinitesimal first-order deformation of r that remains trianguline and such that the weights of the characters don't change, then it stays de Rham. "Trianguline deformations have two parameters, the weights of the δ_is and the de Rham direction, and those are the only two things we can do."

Proof. Let $[\varphi] \in \ker(H^1_{tr,\underline{\delta}}(G_K, \mathrm{ad}\, r) \to T_{\underline{\delta}}\mathscr{W}_{K,n})$. Then $(1 + \epsilon\varphi)r$ is trianguline with parameters $\tilde{\delta}_i$ such that $\tilde{\delta}_i|_{\mathcal{O}_K^\times} = \delta_i|_{\mathcal{O}_K^\times}$. Each $R_{L[\epsilon]/(\epsilon^2)}(\pi_K)(\tilde{\delta}_i)$ is de Rham and the weights are increasing. But we have seen that extensions of de Rham representations where the weights are increasing are de Rham, so $(1 + \epsilon\varphi)r$ is de Rham, so $[\varphi] \in H^1_g$. □

So it's good to know when things are noncritical. Fortunately, in the de Rham situation, we can think about it in terms of filtered WD reps instead of (φ, Γ)-modules.

Let $r : G_K \to GL_n(L)$ be de Rham (and L sufficiently large). Then triangulations of $M_{rig}(r) = M_{rig}(D_{pst}(r))$ correspond to filtrations $\mathrm{Fil}_\bullet$ on $WD(r)$ of WD reps such that $\mathrm{gr}_i\, WD(r)$ is 1-dimensional for $i = 1, \dots, n$. So r is trianguline if and only if $WD(r)^{ss} = \chi_1 \oplus \cdots \oplus \chi_n$. Assume for simplicity $\chi_i \neq \chi_j$ for all $i \neq j$ (though this isn't strictly necessary). Triangulations correspond to orderings of the characters χ_i such that for all i, $\chi_1 \oplus \cdots \oplus \chi_i$ is a sub-WD-rep (invariant under N). If $N = 0$ there are $n!$ triangulations.

What are the δ_is? We need to change the χ_is by powers of embeddings of K into L determined by where the χ_i-eigenspaces sit with respect to the filtration. It turns out $M_{rig}(r)$ is trianguline with parameters $\delta_i = \chi_i \prod_\tau \tau^{-k_{\tau,i}}$, where $HT_\tau(r) = \{k_{\tau,1} < \cdots < k_{\tau,n}\}$, unless

$$\chi_1 \oplus \cdots \oplus \chi_i \cap \mathrm{Fil}_\tau^{k_{\tau,i+1}} \neq (0).$$

The first thing is i-dimensional and the second thing is $(n-i)$-dimensional, and they're in an n-dimensional subspace, so this typically won't happen. On the other hand it is always the case that

$$\chi_1 \oplus \cdots \oplus \chi_i \cap \mathrm{Fil}_\tau^{k_{\tau,i}} \neq (0)$$

since these have dimensions i and $n+1-i$. So generically the filtration on the ith graded piece will see the $k_{\tau,i}$-step but not the $k_{\tau,i+1}$-step.

In this case (when $\delta_i = \chi_i \prod_\tau \tau^{-k_{\tau,i}}$ for all i), we have $wt_\tau(\delta_i) = k_{\tau,i}$, and $\underline{\delta}$ is regular. Why? For $j > i$, we want to check whether

$$\frac{\chi_i \prod \tau^{-k_{\tau,i}}}{\chi_j \prod \tau^{-k_{\tau,j}}} \stackrel{?}{=} \prod_\tau \tau^{m_\tau}$$

for $m_\tau \in \mathbb{Z}_{\geq 0}$, i.e.

$$\chi_i / \chi_j \stackrel{?}{=} \prod_\tau \tau^{m_\tau + k_{\tau,1} - k_{\tau,j}},$$

but the LHS has open kernel and the RHS can only have open kernel if it is trivial, so this is only possible if $m_\tau + k_{\tau,i} - k_{\tau,j} = 0$ for all τ and $\chi_i = \chi_j$. But we assumed that $\chi_i \neq \chi_j$.

Also in this case $wt_\tau(\delta_1) < wt_\tau(\delta_2) < \cdots$, i.e. $\underline{\delta}$ is noncritical, since we defined the $k_{\tau,i}$s to be in increasing order.

The trouble is if you're just handed a representation it's very hard to know where the filtration is. If you have an automorphic form giving rise to a Galois representation, you can read off the characters χ_i and the HT numbers from the WD rep, but there's no information from the automorphic side that tells us where the filtration is. So it would be nice to have some criterion that guarantees non-criticality.

If $(\chi_1 \oplus \cdots \oplus \chi_i) \cap \mathrm{Fil}_\tau^{k_{\tau,i+1}} \neq (0)$ for some i, τ, choose the smallest i for which this fails. We have $(\chi_1 \oplus \cdots \oplus \chi_i) \subset WD(r)$, which we know is admissible. So

$$v_p((\chi_1, \dots, \chi_i)(\mathrm{Frob}_p)) \geq \sum_\sigma \sum_{j \leq i} k_{\sigma,j} + (k_{\tau,i+1} - k_{\tau,i})$$

(the first sum is what we would expect in the generic case; if that fails, the submodule must have a larger HT weight, giving the second term). Conversely, if

$$v_p((\chi_1, \ldots, \chi_i)(\mathrm{Frob}_p)) < \sum_\sigma \sum_{j \le i} k_{\sigma,j} + (k_{\tau,i+1} - k_{\tau,i})$$

for all i, τ, then $M_{rig}(r)$ must be trianguline with regular and noncritical parameter $\delta_i = \chi_i \prod_\tau \tau^{-k_{\tau,i}}$. This numerical criterion is much easier to verify than the conclusion, though even if it fails the conclusion is likely to be true. (UPDATE: these expressions are slightly off, see next lecture for correct version.)

Next time, we want to parametrize the general trianguline representation and understand its geometry (the "trianguline variety").

Philosophically, the trianguline condition is roughly a way to generalize the de Rham condition to non-integral HT numbers—to vary the weight but keep the other properties. It runs into the following problems. It doesn't capture all de Rham representations, only those that become semistable over an abelian extension, but that's not so serious since you could say "potentially trianguline". It also adds a finite amount of extra information (the ordering). Finally, it captures more than de Rham representations even at integers, but there are additional criteria you can sometimes check to make them de Rham. In the situation we looked at above you start with a noncritical de Rham point and deform it, keeping it trianguline and keeping the HT numbers the same, and then it remains de Rham.

14 March 2: trianguline varieties.

14.1 Recap, corrections, and loose ends

We were looking at the rigid variety $\mathscr{C}_{K,n}$ parametrizing characters $\underline{\delta} = (\delta_1, \ldots, \delta_n) : K^\times \to \cdots$, which maps to $\mathscr{W}_{K,n}$ parametrizing characters of $(\mathcal{O}_K^\times)^n$. We called $\underline{\delta} \in \mathscr{C}_{K,n}$ regular if for $i < j$, $\delta_i/\delta_j \neq \prod_{\tau: K \hookrightarrow \overline{L}} \tau^{m_\tau}$ for $m_\tau \in \mathbb{Z}_{\ge 0}$. We called $\underline{\delta}$ noncritical if $wt_\tau(\delta) \in \mathbb{Z}$ for all τ, i, and

$$wt_\tau(\delta_1) < wt_2(\delta_2) < \cdots < wt_\tau(\delta_n)$$

for all τ. It is *false* that this implies regularity, we said that last time but that was a mistake.

We call $\underline{\delta}$ numerically noncritical if $wt_\tau(\delta_i) \in \mathbb{Z}$ for all τ, i and

$$\max\{0, v_p((\delta_1 \cdots \delta_i)(p))\} < wt_\tau(\delta_{i+1}) - wt_\tau(\delta_i).$$

Obviously numerically noncritical implies noncritical.

If $r : G_K \to GL_n(L)$ is de Rham and $M_{rig}(r)$ is trianguline with parameter $\underline{\delta}$ which is regular and noncritical (we might have just said noncritical before, but we need both), then

$$\ker(H^1_{tr,\underline{\delta}}(G_K, \mathrm{ad}\, r) \to T_{\underline{\delta}} \mathscr{W}_{K,n}) \subset H^1_g(G_K, \mathrm{ad}\, r)$$

where $H^1_g(G_K, \mathrm{ad}\, r)$ parametrizes de Rham deformations.

If $r : G_K \to GL_n(L)$ is de Rham, we can directly define a triangulation of $WD(r)$ to be an increasing filtration Fil_i on $WD(r)$ by WD-submodules (not to be confused with the decreasing Hodge filtrations coming from the de Rham nature of r, which are not filtrations of WD-submodules) such that $\mathrm{Fil}_0 = (0)$, $\mathrm{Fil}_n = WD(r)$, and $\mathrm{gr}_i \cong L(\chi_i)$ for some character $\chi_i : W_K \to L^\times$ (equivalently by class field theory a character $K^\times \to L^\times$ that is smooth, i.e. has open kernel). We say that r has parameter $\underline{\chi} = (\chi_1, \ldots, \chi_n)$.

Triangulations of $WD(r)$ are in bijection with triangulations of $M_{rig}(r)$ (without specifying how the characters match up).

We call $\underline{\chi}$ regular in the above setting if $\chi_i \neq \chi_j$ for all $i \neq j$. Regular for a triangulation of $WD(r)$ implies regular for a triangulation of $M_{rig}(r)$, but the converse is not true, because δ_i differs from χ_i by a product of τs to integer powers.

Let $HT_\tau(r) = \{k_{\tau,1} \leq \cdots \leq k_{\tau,n}\}$. We call $\mathrm{Fil}_\bullet$ on $WD(r)$ noncritical if it is in general position with respect to the Hodge filtrations, that is, for all τ, i, we have $k_{\tau,i} < k_{\tau,i+1}$ and

$$\mathrm{Fil}_i \cap \mathrm{Fil}_\tau^{k_{\tau,i+1}} = (0).$$

$\mathrm{Fil}_\bullet WD(r)$ being noncritical implies that the corresponding triangulation of $M_{rig}(r)$ is noncritical and has parameter $\underline{\delta}$ with $\delta_i = \chi_i \prod_\tau \tau^{-k_{\tau,i}}$ (instead of the powers $k_{\tau,i}$ being weirdly permuted). (Conversely, if $(M_{rig}(r), \underline{\delta})$ is a noncritical triangulation and $wt_\tau(\delta_i)$ are all different [for a fixed τ but varying i], then the corresponding triangulation of $WD(r)$ is noncritical.)

In the noncritical case, $\underline{\chi}$ is regular if and only if $\underline{\delta}$ is regular.

We call $(r, \underline{\chi})$ numerically noncritical if for all τ, i,

$$v((\chi_1 \cdots \chi_i)(p)) - \sum_\tau \sum_{j \leq i} k_{\tau,j} < k_{\tau,i+1} - k_{\tau,i}.$$

Last time, we mistakenly put Frob_K instead of p in $v((\chi_1 \cdots \chi_i)(p))$, i.e. a uniformizer of $\mathcal{O}_K$, but if we wanted to do that we should have multiplied by the ramification index of $K/\mathbb{Q}_p$. Better just to put p. This implies that $(r, \underline{\chi})$ is noncritical, and is equivalent to $(M_{rig}(r), \underline{\delta})$ being numerically noncritical.

14.2 Local Galois deformations

Fix $L/\mathbb{Q}_p$ finite, $\mathcal{O} = \mathcal{O}_K$, $\mathcal{O}_L/\lambda = \mathbb{F}$ (a finite field of characteristic p), and $\overline{r} : G_K \to GL_n(\mathbb{F})$ absolutely irreducible. (We are actually going to need to know what to do when it's reducible, but we'll discuss that later; the irreducible case is more intuitive.) There is a universal lifting

$$r^{univ} : G_K \to GL_n(R^{univ}),$$

specifically the universal deformation to a complete noetherian local $\mathcal{O}$-algebra with residue field $\mathbb{F}$, meaning that

$$r^{univ} \pmod{\mathfrak{m}_{R^{univ}}} = \overline{r}$$

and that given any $r : G_K \to GL_n(R)$ where R is a complete noetherian local $\mathcal{O}$-algebra with residue field $\mathbb{F}$, there is a unique $f : R^{univ} \to R$ such that $f(r^{univ}) \sim r$, meaning they are conjugate by an element of $\ker(GL_n(R) \to GL_n(\mathbb{F}))$.

The trianguline condition is optimized to work in rigid analytic families, but c.n.l.etc. $\mathcal{O}$-algebras R are fundamentally not rigid analytic objects (e.g. not affinoid algebras after inverting p), so we will forget some information and just take the rigid analytic generic fiber

$$X_{\overline{r}}^{univ} = (\mathrm{Spf}\, R^{univ})^{an}.$$

(For example, the rigid generic fiber of a power series ring in one variable is a rigid analytic open unit disc.) Points of $X_{\overline{r}}^{univ}$ are in bijection with maximal ideals of $R^{univ}[1/p]$. If $x \in X_{\overline{r}}^{univ}$, we have (by general rigid geometry facts)

$$\mathscr{O}^{\wedge}_{X_{\overline{r}}^{univ},x} = R^{univ}[1/p]^{\wedge}_{\mathfrak{m}_x},$$

the universal deformation ring for $r_x : G_K \to GL_n(R^{univ}[1/p]/\mathfrak{m}_x)$. So the completion of the structure sheaf at a point has a Galois theoretic meaning. Pushing forward gives

$$r^{univ} : G_K \to GL_n(\mathscr{O}_{X_{\overline{r}}^{univ}}).$$

Now let's find the trianguline locus. Inside $X_{\overline{r}}^{univ} \times \mathscr{C}_{K,n}^{reg}$, define $\mathscr{T}^{reg}$ to be the subset of points $(x, \underline{\delta})$ such that r_x is trianguline with parameter $\underline{\delta}$ (uniquely since $\underline{\delta}$ is regular). Let $\mathscr{T}$ be the Zariski closure of $\mathscr{T}^{reg}$ in $X_{\overline{r}}^{univ} \times \mathscr{C}_{K,n}$. We call $\mathscr{T}$ the local trianguline variety.

Recall our general theorem about interpolating triangulations: there is a ("blow-up") map $\pi : \tilde{\mathscr{T}} \to \mathscr{T}$ that is proper and birational such that $\pi^* M_{rig}(r^{univ})$ has a unique filtration Fil_i by (φ, Γ_K)-stable $R_{\tilde{\mathscr{T}}}(\pi_K)$-submodules such that

1. $Y = \{y \in \tilde{\mathscr{T}} \mid (\mathrm{Fil}_i\, \pi^* M_{rig}(r^{univ}))_y$ is not strictly trianguline with parameter $\delta_{\underline{y}}\}$ is Zariski closed and disjoint from $\pi^{-1}\mathscr{T}^{reg}$.

2. there is a (φ, Γ_K)-equivariant $\mathrm{gr}_i\, \pi^* M_{rig}(r^{univ}) \hookrightarrow R_{\tilde{\mathscr{T}}}(\pi_K)(\delta_i) \otimes \mathscr{L}_i$, where $\mathscr{L}_i$ is a line bundle on $\tilde{\mathscr{T}}$, such that the cokernel is killed by a power of t and supported on Y.

We will see that

$$(\mathscr{T} - \pi Y) \cap (X_{\overline{r}}^{univ} \times \cap_{K,n}^{reg}) = \mathscr{T}^{reg}.$$

Note that the LHS is Zariski open in $\mathscr{T}$, because π is proper and Y is closed, so πY is closed, so $\mathscr{T} - \pi Y$ is open, and also $\mathscr{C}_{K,n}^{reg}$ is open in $\mathscr{C}_{K,n}$. Consequently $\mathscr{T}^{reg}$ is not just a set but a rigid analytic subspace. To get the equality, the $\supset$ inclusion is clear because $\mathscr{T}^{reg}$ was defined as a subset of $X_{\overline{r}}^{univ} \times \cap_{K,n}^{reg}$ and we know πY is disjoint from it. For the $\subset$ inclusion, suppose given a point in the LHS. Then its preimage in $\tilde{\mathscr{T}}$ is not in Y, so the M_{rig} of the preimage will have a triangulation with the correct parameters, so the associated Galois representation r will be in $\mathscr{T}^{reg}$.

Now for the tangent space, or even the entire formal completion.

Proposition 14.2.1. *We have a surjection*

$$R^{univ}_{r_x, tr, \underline{\delta}} \twoheadrightarrow \mathscr{O}^{\wedge}_{\mathscr{T}^{reg},(x,\underline{\delta})},$$

where the source is the universal deformation ring for deformations of r_x which are trianguline with parameter deforming $\underline{\delta}$. (Richard strongly suspects this is an isomorphism.)

What is $R^{univ}_{r_x,tr,\underline{\delta}}$? We have discussed the relative representability of trianguline deformations of r_x with parameter deforming $\underline{\delta}$ over artinian rings, so those give quotients of the universal deformation ring, and passing to the limit gives you complete noetherian local algebras, which is possible because we proved that $tr, \underline{\delta}$ is a closed condition.

Proof. Since $\mathscr{T}^{reg}$ is locally closed in $X^{univ}_r \times \mathscr{C}^{reg}_{K,n}$, we have a surjection

$$(\mathscr{O}_{X^{univ}_r \times \mathscr{C}^{reg}_{K,n}})^\wedge_{(x,\underline{\delta})} \twoheadrightarrow \mathscr{O}^\wedge_{\mathscr{T}^{reg},(x,\underline{\delta})}.$$

We also have a natural projection

$$(\mathscr{O}_{X^{univ}_r \times \mathscr{C}^{reg}_{K,n}})^\wedge_{(x,\underline{\delta})} \twoheadrightarrow \mathscr{O}^\wedge_{X^{univ}_r,x} = R^{univ}_{\overline{r}}[1/p]^\wedge_x = R^{univ}_{r_x} \twoheadrightarrow R^{univ}_{r_x,tr,\underline{\delta}}.$$

It suffices to show that if $f \in (\mathscr{O}_{X^{univ}_r \times \mathscr{C}^{reg}_{K,n}})^\wedge_{(x,\underline{\delta})}$ does not go to 0 in $\mathscr{O}^\wedge_{\mathscr{T}^{reg},(x,\underline{\delta})}$, then it also does not go to 0 in $R^{univ}_{r_x,tr,\underline{\delta}}$. Take such an f. Then f is nonzero in the completion of $\tilde{\mathscr{T}}$ at the fiber over $(x, \underline{\delta})$, so nonzero at some point in that fiber, so there is $\tilde{x} \in \tilde{\mathscr{T}}$ mapping down to $(x, \underline{\delta}) \in \mathscr{T}$ such that $f \neq 0$ under the map

$$\mathscr{O}^\wedge_{\mathscr{T}^{reg},(x,\underline{\delta})} \to \mathscr{O}^\wedge_{\tilde{\mathscr{T}},\tilde{x}}.$$

(Actually, Richard suspects that π doesn't blow up at points in $\mathscr{T}^{reg}$ so this isn't necessary, but doesn't know for sure.) But $\tilde{\mathscr{T}}$ has a global triangulation, hence so does its completions, so we get a map

$$R^{univ}_{r_x,tr,\underline{\delta}} \to \mathscr{O}^\wedge_{\tilde{\mathscr{T}},\tilde{x}}.$$

These maps commute, so we conclude that $f \neq 0$ in $R^{univ}_{r_x,tr,\underline{\delta}}$, as desired. ☐

Passing to tangent spaces, we have a surjection

$$H^1_{tr,\underline{\delta}}(G_K, \operatorname{ad} r_x) \twoheadrightarrow T_{(x,\underline{\delta})}\mathscr{T}^{reg}$$

(which Richard again suspects should be an isomorphism).

In summary, we have a local trianguline variety that essentially parametrizes trianguline deformations of a given $\overline{r}$, except with errors in the parameters at a bad locus.

14.3 Global deformations

Let F be an imaginary CM field with maximal totally real subfield F^+ such that p splits in F/F^+ (by which we mean that each prime dividing p in F^+ splits in F). Previously we saw that Galois representations associated to automorphic representations have target in the expanded group

$$\mathscr{G}_n = (GL_n \times GL_1) \rtimes \{1, j\}$$

where $j(g, \lambda)j^{-1} = (\lambda {}^t g^{-1}, \lambda)$, which came with the natural map

$$\begin{aligned} \nu : \mathscr{G}_n &\to \mathbb{G}_m \\ (g, \lambda) &\mapsto \lambda \\ j &\mapsto -1. \end{aligned}$$

Let $\overline{r} : G_{F^+} \to \mathscr{G}_n(\mathbb{F})$ be such that under the surjections $G_{F^+} \twoheadrightarrow \mathrm{Gal}(F/F^+)$ and $\mathscr{G}_n(\mathbb{F}) \twoheadrightarrow \{1, j\}$, we get an isomorphism $\mathrm{Gal}(F/F^+) \xrightarrow{\sim} \{1, j\}$. Also assume

$$\nu \circ \overline{r} = \epsilon_L^{1-n} \delta_{F/F^+}^n$$

where $\delta_{F/F^+} : \mathrm{Gal}(F/F^+) \xrightarrow{\sim} \{\pm 1\}$. We will write $\overline{r}^0 : G_F \to GL_n(\mathbb{F})$ for the restriction of $\overline{r}$ to G_F. Let S be a finite set of places of F^+ including all the ones above p and all the ones where $\overline{r}$ is ramified.

We have a global universal deformation

$$r_S^{univ} : G_{F^+,S} \to \mathscr{G}_n(R_{S,\overline{r}}^{univ})$$

where $G_{F^+,S} := \mathrm{Gal}(F_S^+/F^+)$ where F_S^+ is the maximal algebraic extension of F^+ unramified outside S, and $R_{S,\overline{r}}^{univ}$ is a complete noetherian local $\mathcal{O}$-algebra with residue field $\mathbb{F}$. We can again define

$$X_{\overline{r},S}^{univ} = (\mathrm{Spf}\, R_{S,\overline{r}}^{univ})^{an},$$

$$r_S^{univ} : G_{F^+,S} \to GL_n(\mathscr{O}_{X_{\overline{r},S}^{univ}}),$$

and

$$\mathscr{C}_{F,n} = \prod_{v|p \text{ prime of } F} \mathscr{C}_{F_v,n} \twoheadrightarrow \mathscr{W}_{F,n} = \prod_{v|p} \mathscr{W}_{F_v,n}.$$

Actually what we really want is the conjugate self-dual locus

$$\mathscr{C}_{F,n}^{csd} \subset \mathscr{C}_{F,n}$$

parametrizing $(\chi_{v,i})$ such that $\chi_{cv,i} = \chi_{v,n+1-i}^{-c}$, and similarly the weight space

$$\mathscr{C}_{F,n}^{csd} \twoheadrightarrow \mathscr{W}_{F,n}^{csd} \subset \mathscr{W}_{F,n}$$

and the regular locus $\mathscr{C}_{F,n}^{csd,reg} \subset \mathscr{C}_{F,n}^{csd}$. As before, let

$$\mathscr{T}^{reg} \subset X_{\overline{r},S}^{univ} \times \mathscr{C}_{F,n}^{csd,reg}$$

be the set of points $(x, \underline{\delta})$ where $r_x^0|_{G_{F_v}}$ is trianguline with parameter $\underline{\delta}_v$. Let $\mathscr{T}$ be the Zariski closure of $\mathscr{T}^{reg}$ in $X_{\overline{r},S}^{univ} \times \mathscr{C}_{F,n}^{csd}$. Again we have a $\pi : \tilde{\mathscr{T}} \to \mathscr{T}$ with the same properties as before (say with bad set Y), an equality

$$\mathscr{T}^{reg} = (\mathscr{T} - \pi Y) \cap (X_{\overline{r},S}^{univ} \times \mathscr{C}_{F,n}^{csd,reg}),$$

and a surjection

$$R_{r_x,S,tr,\underline{\delta}}^{univ} \twoheadrightarrow \mathscr{O}_{\mathscr{T}^{reg},(x,\underline{\delta})}^{\wedge}$$

which is probably an isomorphism. The LHS parametrizes deformations r of r_x such that r^0 is trianguline at all $v|p$ with parameter deforming $\underline{\delta}$.

The infinitesimal deformations of r_x unramified outside S and trianguline at all $v|p$ with parameter deforming $\underline{\delta}$ are

$$D_{r_x,S,tr,\underline{\delta}}(k(x)[\epsilon]/(\epsilon^2)) \twoheadrightarrow T_{(x,\underline{\delta})}\mathscr{T}^{reg} \to T_{\underline{\delta}}\mathscr{W}_{F,n}.$$

This can also be written as

$$D_{r_x,S}(k(x)[\epsilon]/(\epsilon^2)) \times_{\bigoplus_{v|p} D_{r(x)^0|_{G_{F_v}}}(k(x)[\epsilon]/(\epsilon^2))} D_{r_x^0|_{G_{F_v}},\underline{\delta}_v,tr}(k(x)[\epsilon]/(\epsilon^2))$$

$$= H^1(G_{F^+,S}, \operatorname{ad} r_x) \times_{\bigoplus_{v|p} H^1(G_{F_v}, \operatorname{ad} r_x^0)} \bigoplus_{v|p} H^1_{tr,\underline{\delta}}(G_{F_v}, \operatorname{ad} r_x^0).$$

$H^1(G_{F^+,S}, \operatorname{ad} r_x)$ is an example of a Selmer group, that is, it is of the form $H^1_{\mathscr{L}_{S,\underline{\delta}}}(G_{F^+}, \operatorname{ad} r_x)$ where $\mathscr{L}_{S,\underline{\delta}}$ is as follows. The terminology is that if G_{F^+} acts on some module $M/\mathbb{Q}_p$, we choose a set $\mathscr{L} = \{L_v\}$ where $L_v \subset H^1(G_{F_v^+}, M)$ for each place v of F^+, such that

$$L_v = H^1(G_{F_v^+}/I_{F_v^+}, M^{I_{F_v^+}})$$

for almost all v (say for $v \notin T$ where T contains places where M is ramified and places above p), and define

$$H^1_{\mathscr{L}}(G_{F^+}, M) = \ker\left(H^1(G_{F^+}, M) \to \bigoplus_v H^1(G_{F_v^+}, M)/L_v \right)$$
$$= \ker\left(H^1(G_{F^+,T}, M) \to \bigoplus_{v\in T} H^1(G_{F_v^+}, M)/L_v \right).$$

In our case we choose

$$L_{S,\underline{\delta},v} = \begin{cases} H^1_{tr,\underline{\delta}_v}(G_{F_v}, \operatorname{ad} r_x^0) & v|p \\ H^1(G_{F_v^+}, \operatorname{ad} r_x) & v \in S, v \nmid p \\ H^1(G_{F_v^+}/I_{F_v^+}, \operatorname{ad} r_x) & v \notin S. \end{cases}$$

(We don't need to take inertia-fixed points in the last case since $\operatorname{ad} r_x$ is unramified at $v \notin S$.)

There are two other important examples of Selmer groups to keep in mind. The local conditions are

$$H^1_f(G_{F_v^+}, M) = \begin{cases} H^1(G_{F_v^+}/I_{F_v^+}, M^{I_{F_v^+}}) & v \nmid p \\ \ker\left(H^1(G_{F_v^+}, M) \to H^1(G_{F_v^+}, M \otimes B_{crys})\right) & v|p \end{cases}$$

$$H^1_g(G_{F_v^+}, M) = \begin{cases} H^1(G_{F_v^+}, M) & v \nmid p \\ \ker\left(H^1(G_{F_v^+}, M) \to H^1(G_{F_v^+}, M \otimes B_{dR})\right) & v|p \end{cases}$$

and they give rise to the Selmer groups $H^1_f(G_{F^+}, M)$ given by

$$\mathscr{L}_f = \left\{ H^1_f(G_{F_v^+}, M) \right\}$$

and $H^1_{S,g}(G_{F^+}, M)$ given by

$$\mathscr{L}_{S,g} = \begin{cases} H^1_f(G_{F_v^+}, M) & v \notin S \\ H^1_g(G_{F_v^+}, M) & v \in S. \end{cases}$$

Next time, we'll relate the tangent space at the global trianguline variety to these Selmer groups and to the local trianguline variety in order to compute its dimension.

15 March 4: eigenvariety setup.

15.1 Selmer groups recap

Last time, we chose F/F^+ an imaginary CM field with totally real subfield and p unramified, and fixed $\overline{r} : G_{F^+} \to \mathscr{G}_n(\mathbb{F})$ where $L/\mathbb{Q}_p$, $\mathcal{O}$, and $\mathbb{F}$ were chosen. We wrote $\overline{r}^0 : G_F \to GL_n(\mathbb{F})$ for the restriction and assumed it was absolutely irreducible. We defined $R^{univ}_{\overline{r},S}$ to be the deformation ring for reps unramified outside S, and wrote

$$X^{univ}_{\overline{r},S} = (\mathrm{Spf}\, R^{univ}_{\overline{r},S})^{an}.$$

This gives rise to $r^{univ} : G_{F^+} \to \mathscr{G}_n(\mathscr{O}_{X^{univ}_{\overline{r},S}})$. We defined a trianguline variety

$$\mathscr{T}^{reg} \overset{\text{dense Zariski open}}{\subset} \mathscr{T} \overset{\text{Zariski closed}}{\subset} X^{univ}_{\overline{r},S} \times \mathscr{C}^{csd}_{F,n}$$

where $\mathscr{T}^{reg}$ consists of pairs $(x, \underline{\delta})$ where r_x is strictly trianguline with parameter $\underline{\delta}$ and $\mathscr{C}^{csd}_{F,n}$ consists of conjugate self-dual characters of $(F_p^\times)^n$. Given $(x, \underline{\delta}) \in \mathscr{T}^{reg}$, we saw that $\mathscr{O}^\wedge_{\mathscr{T}^{reg},(x,\underline{\delta})}$ is the universal deformation ring for trianguline deformations of r_x with parameter deforming $\underline{\delta}$, and that

$$T_{(x,\underline{\delta})}\mathscr{T}^{reg} = H^1_{tr,\underline{\delta}}(G_{F^+,S}, \mathrm{ad}\, r_x)$$

where the RHS is a Selmer group. We defined Selmer groups in general: you choose Selmer conditions

$$\mathscr{L} = \{L_v \subset H^1(G_{F^+,v}, M)\}$$

which have to be the subset of unramified classes for all but finitely many v, and define the Selmer group $H^1_{\mathscr{L}}(G_{F^+}, M)$ to be the global classes in $H^1(G_{F^+}, M)$ which everywhere locally lie in $L_v \subset H^1(G_{F^+,v}, M)$. For $H^1_{tr,\underline{\delta}}(G_{F^+,S}, \mathrm{ad}\, r_x)$, we took

$$L_v = \begin{cases} H^1_f(G_{F^+}, \mathrm{ad}\, r_x) & v \notin S \\ H^1_g(G_{F^+}, \mathrm{ad}\, r_x) & v \in S, v \nmid p \\ H^1_{tr,\underline{\delta}_v}(G_{F_v^+}, \mathrm{ad}\, r_x) & v | p. \end{cases}$$

We also wrote down the more familiar versions $H^1_f(G_{F^+}, \mathrm{ad}\, r_x)$, where $L_v = H^1_f(G_{F_v^+}, \mathrm{ad}\, r_x)$ for all v, and $H^1_S(G_{F^+}, \mathrm{ad}\, r_x)$, where L_v is H^1_f for $v \notin S$ and H^1_g for $v \in S$. Remember that H^1_g is everything if $v \nmid p$ and the de Rham classes if $v|p$.

If r_x is de Rham, $\underline{\delta}$ is noncritical, and $(x, \underline{\delta}) \in \mathscr{T}^{reg}$, we saw that

$$H^1_S(G_{F^+}, \mathrm{ad}\, r_x) \twoheadrightarrow \ker\left(H^1_{S,tr,\underline{\delta}}(G_{F^+}, \mathrm{ad}\, r_x) \to T_{\underline{\delta}}\mathscr{W}^{csd}_{F,n}\right).$$

This was because $H^1_{S,tr,\underline{\delta}}(G_{F^+}, \mathrm{ad}\, r_x)$ consists of the classes that locally lie in $H^1_{tr,\underline{\delta}_v}$, and the map to weight space factors through the maps to local cohomology, and we know the statement locally.

We want to look at the difference between H^1_f and H^1_S. It's a general fact that given $M/\mathbb{Q}_l$, we have

$$\dim_{\mathbb{Q}_l} H^1_g(G_{F_v^+}, M)/H^1_f(G_{F_v^+}, M) = \dim_{\mathbb{Q}_l} WD(M^\vee(1))^{W_{F_v^+}, N=0}.$$

($M^\vee(1)$ is called the Tate dual—dualize and twist by the cyclotomic character; it doesn't matter whether you dualize before or after taking WD.) For $v \nmid p$, this is an elementary computation using the restriction-inflation long exact sequence, which tells us that

$$H^1(G_{F_v^+}, M)/H^1\left(G_{F_v^+}/I_{F_v^+}, M^{I_{F_v^+}}\right) \hookrightarrow H^1(I_{F_v^+}, M)^{G_{F_v^+}}$$

and since the next thing is $H^2(G^+_{F_v}/I_{F_v^+}, \cdots) = H^2(\hat{\mathbb{Z}}, \cdots)$ which is 0, this is actually an isomorphism. Now we have

$$H^1(I_{F_v^+}, M)^{G_{F_v^+}} = M_{I_{F_v^+}}(-1)^{G_{F_v^+}}$$

(because we just need to look at the p-power quotient, which is just a copy of $\mathbb{Z}_p$, and $H^1(\mathbb{Z}_p, \cdots)$ is just coinvariants, and then you get a (-1) from the action of Frobenius on tame inertia). Dualizing this (which doesn't change the dimension) gives

$$\left(M^\vee(1)^{I^+_{F_v}}\right)_{(\mathrm{Frob}_v - 1)}$$

(note that the cyclotomic character is unramified because $v \nmid p$). Since $M(1)^{I^+_{F_v}}$ is a finite-dimensional vector space, the dimension of the kernel of the endomorphism $\mathrm{Frob}_v - 1$ equals the dimension of the cokernel, so

$$\dim H^1_g/H^1_f = \dim\left(M^\vee(1)^{I_{F_v^+}}\right)^{\mathrm{Frob}_v=1} = (M^\vee(1))^{G_{F_v^+}} \cong WD(M^\vee(1))^{W_{F_v^+}, N=0}$$

where the last equality is the standard transformation from the Galois to the WD action. For $v|p$, you need B_{crys} and B_{dR}—look it up in the Bloch-Kato paper on Tamagawa numbers ([2]).

15.2 Newton-Thorne propagation

Theorem 15.2.1 (Newton-Thorne)**.** *Let F be an imaginary CM field and π a PRAC automorphic representation of $GL_n(\mathbb{A}_F)$. Fix $i : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$. Suppose that $r_{p,i}(\pi)(G_{F(\zeta_{p^\infty})})$ is "enormous". Then*

$$H^1_f(G_{F^+}, \mathrm{ad}\, r_p(\pi)) = (0).$$

Here we say that a subgroup $H \subset GL(V)$ is enormous if, considering the action of H by conjugation on $\mathrm{End}(V)$, for all nonzero H-invariant subspaces $W \subset \mathrm{End}(V)$, there is $h \in H$ with distinct eigenvalues and an eigenvalue α of h such that if $e_{h,\alpha}$ is the idempotent in $\mathrm{End}(V)$ which projects h-equivariantly to the α-eigenline of h, then $\mathrm{tr}(e_{h,\alpha}W) \neq (0)$.

This kind of "enormous" condition is very common in automorphy lifting because to use Chebotarev density you want to know that H has a lot of elements. It is generally not a hard condition to check because it can be checked upon passing to the Zariski closure.

The way to prove this theorem is by the usual automorphy lifting arguments. The typical output of such an argument is "$R = T$", and the Rs that come up are H^1_fs: you look at de Rham lifts of fixed HT numbers/action of inertia, and the tangent space of the universal deformation rings is related to H^1_f. When you show that the deformation ring is equal to a Hecke algebra, since the Hecke algebra after inverting p is just a product of fields, its tangent space vanishes, and you get $H^1_f = 0$ as a byproduct.

What's striking about this particular theorem is the lack of conditions. The "enormous" condition is only on the image of the l-adic representation. Usually you require such a thing on the residual representation, which is much stronger and harder to check. Here, for example, the residual representation can be reducible, which is traditionally hard to deal with.

The way it is done is a recently popular argument, which started with Khare and Thorne, where you start the $R = T$ argument integrally, but once you patch things at infinite level, you invert p, which makes some difficulties go away. Then you descend from the thing at infinite level with p inverted. The bad thing about this is it only gives you $R = T$ after inverting p, and usually that is useless because now T is a product of fields so knowing one point is automorphic only gives you information about that one point (whereas for Taylor-Wiles you want to make a connectedness argument). But since the Hecke algebra is reduced, you do get information about the Selmer group in characteristic 0. So with this approach you've learned nothing about the Galois deformation ring except maybe that it's reduced, but you've picked up a surprising amount of information about the Selmer group. (We won't go further into this argument—it would be an entire course to itself...)

Having $H^1_f(G_{F^+}, \operatorname{ad} r_p(\pi)) = (0)$ implies that $H^1_S(G_{F^+}, \operatorname{ad} r_p(\pi)) = 0$. This is because the difference between the two is bounded by $\dim H^1_f/H^1_g$ for $v \in S$, i.e. using our previous calculation by

$$\sum_{v\in S} \dim \operatorname{Hom}_{WD}\left(WD(r_p(\pi)|_{G_{F_v}}), WD(r_p(\pi)|_{G_{F_v}})(1)\right) = (0)$$

because we're in characteristic 0 and these have different weights: $WD(r_p(\pi)|_{G_{F_v}})$ is pure of some weight and $WD(r_p(\pi)|_{G_{F_v}})(1)$ is therefore pure of a different weight.

Corollary 15.2.2. *Let F be an imaginary CM field and assume p splits in F/F^+. Let π be a PRAC automorphic representation of $GL_n(\mathbb{A}_F)$ such that $r_p(\pi)(G_{F(\zeta_{p^\infty})})$ is enormous. Assume that for all $v|p$, $r_p(\pi)|_{G_{F_v}}$ is trianguline with regular noncritical parameter $\underline{\delta}$.*

Then the map $H^1_{S,tr,\underline{\delta}}(G_{F^+}, \operatorname{ad} r_p(\pi)) \to T_{\underline{\delta}}\mathscr{W}^{csd}_{F,n}$ is an injection (because by the theorem the kernel vanishes), or equivalently we have an injection

$$T_{(r_p(\pi),\underline{\delta})}\mathscr{T} \hookrightarrow T_{\underline{\delta}}\mathscr{W}^{csd}_{F,n}.$$

That is, the trianguline variety is small.

Here is our plan. We are going to construct the global trianguline variety (eigenvariety) $\mathscr{E}$ by automorphic means, and get an embedding $\mathscr{E} \hookrightarrow \mathscr{T}$. Remember that $\mathscr{T}$ has a Galois construction. Both live over $\mathscr{W}$ and $\mathscr{E} \to \mathscr{W}$ is flat and same-dimensional. But we already know that $\mathscr{T}$ has dimension no greater than $\mathscr{W}$ at a suitable point as in the corollary, since $\mathscr{W}$ is smooth and so has tangent space equal to dimension. So any component of $\mathscr{T}$ which contains an automorphic point as in the corollary is in the image of $\mathscr{E}$. So a large union of components in $\mathscr{T}$ are already in $\mathscr{E}$. This means we can propagate automorphy: knowing a few points in $\mathscr{T}$ are automorphic, we deduce that many components are automorphic, hence many Galois representations are automorphic. (To be clear, this doesn't rule out the possibility that there are components of $\mathscr{T}$ with no automorphic points. But it's close to saying that if a component has an automorphic point then the whole component is automorphic.)

What makes this setting different from normal Taylor-Wiles is that in the rigid world you've already inverted p but you still have some connectedness. So you can use the usual argument to calculate the tangent space, but because the result has p inverted, you get stuck with much fewer conditions than usual.

Richard would not be totally surprised if you didn't have to start with a classical automorphic point, and could instead start at any point in $\mathscr{E}$ and get that the corresponding point in $\mathscr{T}$ has tangent space bounded by the dimension of weight space. It's natural to conjecture that the global trianguline variety is equal to the eigenvariety. Certainly when you have a Shimura variety you expect $\mathscr{E}, \mathscr{T}$ to have the same dimension as $\mathscr{W}$ (if not, $\mathscr{E}, \mathscr{T}$ may have smaller dimension). You could get $R = T$ integrally up to a finite error and bound it as you go to ∞. But if there were a component of $\mathscr{T}$ with no automorphic points, Richard wouldn't know how to bound its dimension (but would still believe it was true).

15.3 p-adic automorphic forms on definite unitary groups

Let G/F^+ be the group

$$G(F^+) = \{g \in GL_n(F) \mid g^T c(g) = \mathrm{id}_n\}.$$

For $v|\infty$, $G(F_v^+) = U(n)$. The irreducible algebraic representations of G are, for each $k \in (\mathbb{Z}^n)^{\mathrm{Hom}(F,\mathbb{C})}$ such that $k_{\tau,1} \geq \cdots \geq k_{\tau,n}$ and $k_{c\tau,i} = -k_{\tau,n+1-i}$, the highest weight irreducible algebraic representation σ_k of $G(F_\infty^+)$. Recall that the automorphic forms on G are

$$\mathscr{A}(G(F^+)\backslash G(\mathbb{A}_{F^+}), \mathbb{C}) = \bigoplus \pi$$

where each $\pi_\infty \cong \sigma_k$ for some k. So

$$\mathscr{A}(G(F^+)\backslash G(\mathbb{A}_{F^+}), \mathbb{C}) = \bigoplus_k \mathscr{A}_k(G(F^+)\backslash G(\mathbb{A}_{F^+}), \mathbb{C}) \otimes \sigma_k$$

where

$$\begin{aligned}&\mathscr{A}_k(G(F^+)\backslash G(\mathbb{A}_{F^+}), \mathbb{C})\\ &= \{\varphi : G(F^+)\backslash G(\mathbb{A}_F) \to \sigma_k^\vee \mid \varphi \text{ smooth}, \varphi(gh) = h^{-1}\varphi(g) \forall g \in G(\mathbb{A}_{F^+}), h \in G(F_\infty^+)\}.\end{aligned}$$

For $U^p \subset G(\mathbb{A}_{F^+}^{\infty,p})$, we can also look at the invariants $\mathscr{A}^{U^p}$ and mod out by U^p on the right everywhere above, so $\mathscr{A}(G(F^+)\backslash G(\mathbb{A}_{F^+})/U^p, \mathbb{C})$, $\mathscr{A}_k(G(F^+)\backslash G(\mathbb{A}_{F^+})/U^p, \mathbb{C})$, etc. We want to create a p-adic analogue of this.

How? We could just do $\mathscr{A}^{cts}(G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)/U^p, L)$ for $L/\mathbb{Q}_p$ algebraic and φ continuous. (Note that putting $\mathbb{A}_{F^+}^\infty$ instead of $\mathbb{A}_{F^+}$ in the middle doesn't change anything because there are no nonconstant continuous functions from an archimedean space to a p-adic field.) Or we could do $\mathscr{A}^{la}(G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)/U^p, L)$ where φ has to be locally analytic on $G(F_p^+)$ (i.e. you can cover $G(F_p^+)$ by charts on which it is analytic). Or we could do $\mathscr{A}^{sm}(G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)/U^p, L)$ where φ has to be locally constant.

Now let $U_p \subset G(F_p^+)$ be an open compact subgroup. We can consider σ_k as a representation of $G(F_p^+)$, since it is algebraic. Then we can consider

$$\mathrm{Hom}_{U_p}\left(\sigma_k, \mathscr{A}^{cts/la}(G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)/U^p, L)\right).$$

This is

$$\{\varphi : G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty) \to \sigma_k^\vee \mid u \in U = U^p U_p \implies \varphi(gu) = u_p \varphi(g)\}.$$

This is the same thing as the previous $\mathscr{A}_k$ except h is at p instead of ∞. If $\tau : L \hookrightarrow \mathbb{C}$ is chosen, then

$$\mathrm{Hom}_{U_p}\left(\sigma_k, \mathscr{A}^{cts/la}(\cdots, L)\right) \otimes_{L,\tau} \mathbb{C} \xrightarrow{\sim} \mathscr{A}_k(G(F^+)\backslash G(\mathbb{A}_{F^+})/U, \mathbb{C}).$$

In particular, if

$$\mathscr{A}(G_{F^+}\backslash G(\mathbb{A}_{F^+})) = \bigoplus \pi,$$

then fixing $\overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$,

$$\mathrm{Hom}_{U_p}\left(\sigma_k, \mathscr{A}^{cts/la}(G(F^+)\backslash G(\mathbb{A}_{F^+})/U^p, \overline{\mathbb{Q}}_p)\right) \cong \bigoplus_{\pi|\pi_\infty = \sigma_k} (\pi^\infty)^U.$$

Let's be more precise. Assume $U^p = \prod U_v \subset G(\mathbb{A}_{F^+}^{\infty,p})$ be an open compact subgroup, so that $U_v = G(\mathscr{O}_{F^+,v})$ for $v \notin S$, where $\#S < \infty$ and S contains all primes above p. We can define the abstract Hecke algebra

$$\mathscr{H}_0^S = \sideset{}{'}\bigotimes_{\substack{v \notin S \\ v \text{ split in } F \\ v \text{ place of } F^+}} \mathcal{O}_L\left[G(\mathscr{O}_{F_v^+})\backslash G(F_v^+)/G(\mathscr{O}_{F_v^+})\right]$$

where the restriction is with respect to the identity elements. (These places are enough by Chebotarev density—the Frobeniuses from these places are dense in the Galois group.) This is commutative, a polynomial algebra in infinitely many variables. $\mathscr{H}_0^S$ acts, firstly, on $\mathscr{A}^{cts}(G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)/U^p, \mathcal{O})^{U_p}$ (this is a finite free $\mathcal{O}$-module; also remember that

$$G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)/U^p$$

is compact so given a function valued in L, some p-power multiple of it is valued in $\mathcal{O}$). Then the action of $\mathscr{H}_0^S$ factors through

$$\mathscr{H}_0^S \twoheadrightarrow \mathbb{T}^S(U^p U_p) \subset \mathrm{End}\left(\mathscr{A}^{cts}(G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)/U^p, \mathcal{O})^{U_p}\right)$$

where $\mathbb{T}^S(U^p U_p)$ is an $\mathcal{O}$-algebra, finite and free as an $\mathcal{O}$-module. $\mathbb{T}^S(U^p U_p)$ then acts on

$$\mathscr{A}^{cts}\left(G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)/U^p, \mathcal{O}/p^N\mathcal{O}\right)^{U_p}$$

for U_p sufficiently small, because you can get this from $\mathscr{A}^{cts}(G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)/U^p, \mathcal{O})^{U_p}$ just by tensoring by $\otimes_{\mathcal{O}}\mathcal{O}/p^N$. Taking limits, we get an action of

$$\varprojlim_{U_p} \mathbb{T}^S(U^p U_p) =: \mathbb{T}^S(U^p)$$

on $\mathscr{A}^{cts}\left(G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)/U^p, \mathcal{O}/p^N\mathcal{O}\right)^{U_p}$, and therefore a continuous action of $\mathbb{T}^S(U^p)$ on

$$\mathscr{A}^{cts}(G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)/U^p, \mathcal{O}) = \varprojlim_N \mathscr{A}^{cts}(G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)/U^p, \mathcal{O}/p^N\mathcal{O})$$

and therefore on $\mathscr{A}^{cts}(G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)/U^p, L)$.

Two facts:

1. The number of maximal ideals of $\mathbb{T}(U^pU_p)$ is bounded independently of U_p (we proved this in a special case last year; the proof is the same). Because $\mathbb{T}(U^pU_p)$ is finite over $\mathcal{O}$, we have
$$\mathbb{T}(U^pU_p) \cong \prod_{\mathfrak{m}} \mathbb{T}(U^pU_p)_{\mathfrak{m}}$$
and by taking inverse limits,
$$\mathbb{T}(U^p) = \prod_{\mathfrak{m}} \mathbb{T}(U^p)_{\mathfrak{m}}$$
(which isn't otherwise obvious because $\mathbb{T}(U^p)$ is *not* finite over $\mathcal{O}$). $\mathbb{T}(U^p)$ is a complete noetherian semilocal $\mathcal{O}$-algebra, and each $\mathbb{T}(U^p)_{\mathfrak{m}}$ is a complete noetherian local $\mathcal{O}$-algebra.

2. There is a continuous (WLOG semisimple) representation $\overline{r}_{\mathfrak{m}}^0 : G_F \to GL_n(k(\mathfrak{m}))$. We call $\mathfrak{m}$ Eisenstein if $\overline{r}_{\mathfrak{m}}^0$ is absolutely reducible. If $\mathfrak{m}$ is non-Eisenstein, $\overline{r}_{\mathfrak{m}}^0$ extends to $\overline{r}_{\mathfrak{m}} : G_{F^+} \to \mathscr{G}_n(k(\mathfrak{m}))$ with multiplier $\epsilon_L^{1-n}\delta_{F/F^+}^n$. We have a surjection
$$R_{\overline{r}_{\mathfrak{m}},S}^{univ} \twoheadrightarrow \mathbb{T}(U^p)_{\mathfrak{m}}$$
because you can show that $\overline{r}_{\mathfrak{m}}$ deforms to the Hecke algebra. (In fact, this is how we show that the RHS is noetherian.)

Next time, we will define the eigenvariety.

16 March 9: finite-slope automorphic forms.

No lecture next Tuesday March 16 because Richard will be "at MIT" on their "visiting committee". We'll add 1-2 lectures during spring break. We will probably finish with the proof of the statement that if one level 1 form has an nth symmetric power lifting then they all do. (Finding the one form uses totally different techniques that have nothing to do with the eigenvariety.)

16.1 Recap and loose ends

Let $\underline{\delta} \in \mathscr{C}_{n,K}$ for $K/\mathbb{Q}_p$ finite. Recall that we say $\underline{\delta}$ is

- regular if for $i < j$ $\delta_i/\delta_j \neq \prod_{\tau:K\hookrightarrow\overline{L}} \tau^{m_\tau}$ for $m_\tau \in \mathbb{Z}_{\geq 0}$,
- noncritical if $wt_\tau(\delta_i) \in \mathbb{Z}$ and $wt_\tau(\delta_1) < \cdots < wt_\tau(\delta_n)$, and

- numerically noncritical if $wt_\tau(\delta_i) \in \mathbb{Z}$ and

$$\max\{0, v_p((\delta_1 \cdots \delta_i)(p))\} < wt_\tau(\delta_{i+1}) - wt_\tau(\delta_i).$$

We call a (φ, Γ)-module M generic if it is trianguline and every triangulation is noncritical.

Example 16.1.1. Let $K = \mathbb{Q}_p$ and $n = 2$. Suppose we have a (φ, Γ)-module M which is trianguline with parameter (δ_1, δ_2). Assume $wt(\delta_1) = 0$ (as you typically get for modular forms—if you don't twist by a character, one HT weight is 0). Suppose M is M_{rig} of the Galois rep for a weight k form with $k \geq 2$. Then $wt(\delta_2) = k - 1$. Since the weights are increasing integers, M is de Rham, so by admissibility, $\delta_1(p), \delta_2(p)$ are integral. Then numerical noncriticality says that $v_p(\delta_1(p)) < k - 1$.

On the other hand $\delta_1(p)$ corresponds to the eigenvalue of U_p, and is one of the two roots of the Hecke polynomial, so we also know that $v_p(\delta_1(p)) + v_p(\delta_2(p)) = k - 1$, implying that $v_p(\delta_1(p)) \leq k-1$. So numerical noncriticality just excludes the case $v_p(\delta_1(p)) = k-1$ (i.e. the case that the form is ordinary and we're picking the non-unit root of the Hecke polynomial).

Triangulations of filtered WD reps (with $HT_\tau = \{k_{\tau,1} \leq \cdots \leq k_{\tau,n}\}$) can also be regular, noncritical (meaning $\mathrm{Fil}_i^{tri} \cap \mathrm{Fil}_\tau^{k_{\tau,i+1}} = (0)$), or numerically noncritical, and filtered WD reps can be generic (trianguline and every triangulation is noncritical). It follows from what we've discussed that

{filtered WD reps with a noncritical triangulation} are the same as
{(φ, Γ)-modules with a noncritical triangulation}
and the definitions of regular, numerically noncritical, and generic match up.

Here are two lemmas indicating why these conditions are important.

Lemma 16.1.2. *Suppose $(x, \underline{\delta}) \in \mathscr{T}$ and $\underline{\delta}$ is numerically noncritical. Then r_x is trianguline with parameter $\underline{\delta}$ (as opposed to some other parameter).*

Lemma 16.1.3. *Suppose $(x, \underline{\delta}) \in \mathscr{T}$ and r_x is generic. Then r_x is trianguline with parameter $\underline{\delta}$ (as opposed to some other parameter).*

(Recall that for regular non-critical characters we can also compute tangent spaces and so on.)

16.2 More on p-adic automorphic forms

Recall that we set a CM field F/F^+ and let $G(F^+) = \{g \in GL_n(F) \mid g^T c(g) = \mathrm{id}_n\}$, so for $v|\infty$, $G(F_v^+) = U(n)$. We defined the space $\mathscr{A}(G(F^+)\backslash G(\mathbb{A}_{F^+}), \mathbb{C}) = \bigoplus \pi$, which is nice because $G(F^+)\backslash G(\mathbb{A}_{F^+})$ is compact. For each π, π_∞ is a representation of $G(F_\infty^+)$, which is a product of compact unitary groups; since it is smooth and continuous it must be algebraic, hence corresponds to some highest weight $\underline{k} = (k_{\tau,i}) \in (\mathbb{Z}^n)_+^{\mathrm{Hom}(F,\mathbb{C}),csd}$, meaning that $k_{\tau,1} \geq \cdots \geq k_{\tau,n}$ and $k_{c\tau,i} = -k_{\tau,n+1-i}$. We called the corresponding representation $\sigma_{\underline{k}}$.

For $U^p \subset G(\mathbb{A}_{F^+}^{\infty,p})$ an open compact subgroup and $L/\mathbb{Q}_p$ finite, we then defined

$$\mathscr{A}^{cts}(G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)/U^p, L)$$

(we don't need the ∞ because continuous at ∞ means constant) and saw that it had an action by

$$\mathbb{T}(U^p) = \varprojlim_{U_p} \mathbb{T}(U^p U_p),$$

the algebra with $\mathcal{O} = \mathcal{O}_L$-coefficients generated by the Hecke operators at good places. It is a Banach space over L, and also has an action by $G(F_p^+)$. Given a compact open U_p, we then considered

$$\mathrm{Hom}_{U_p}(\sigma_{\underline{k}}, \mathscr{A}^{cts}(G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)/U^p, L)) \otimes_{L,\tau} \mathbb{C}$$

since $\sigma_{\underline{k}}$ is an algebraic representation of G, hence makes sense as a representation of $G(F_p^+)$. By definition this is

$$\{\varphi : G(F^+)\backslash G(\mathbb{A}_{F^+})/G(F_\infty^+)U^p \to \sigma_{\underline{k}}^\vee(\mathbb{C}) \mid \varphi(gu) = \tau(u^{-1})\varphi(g) \forall u \in U_p\}$$

which is isomorphic to

$$\{\Phi : G(F^+)\backslash G(\mathbb{A}_{F^+})/U^p U_p \to \sigma_{\underline{k}}^\vee(\mathbb{C}) \mid \Phi(gh) = h^{-1}\Phi(g) \forall h \in G(F_\infty^+)\}$$

via $\varphi \mapsto (g \mapsto g_\infty^{-1}\tau(g_p)f(g))$, and the latter can be written as

$$\mathrm{Hom}_{G(F_\infty^+)}(\sigma_{\underline{k}}, \mathscr{A}(G(F^+)\backslash G(\mathbb{A}_{F^+})/U^p U_p, \mathbb{C})) = \bigoplus_{\pi : \pi_\infty \cong \sigma_{\underline{k}}} (\pi^\infty)^{U^p U_p}.$$

That is, the vectors in the p-adic space that are locally algebraic (as in they locally transform according to an algebraic representation of $G(F_p^+)$) correspond to the ones in the space of classical automorphic forms. The continuous p-adic automorphic forms do not split as a direct summand of irreducible representations as the classical ones do because we don't have the strong smoothness hypotheses, but they have some structure: if we write

$$\mathscr{A}_{\underline{k}}(G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)/U^p, L) = \varinjlim_{U_p} \mathrm{Hom}_{U_p}(\sigma_{\underline{k}}, \mathscr{A}^{cts}(G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)/U^p, L))$$

then this has an action of $\mathbb{T}(U^p)$ and also an action of $G(F_p^+)$ which is smooth (stabilizers of vectors are open), and we get an embedding

$$\bigoplus_{\underline{k}} \mathscr{A}_{\underline{k}}(G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)/U^p, L) \otimes \sigma_{\underline{k}} \hookrightarrow \mathscr{A}^{cts}(G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)/U^p, L)$$

which has dense (but no longer surjective) image. The LHS is called the space of locally algebraic elements of the RHS. They recover the classical representations, in that

$$\mathscr{A}_{\underline{k}}(G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)/U^p, L) \otimes_{L,\tau} \mathbb{C} \cong \bigoplus_{\pi \mid \pi_\infty \cong \sigma_{\underline{k}}} \left((\pi^p)^{U^p} \otimes \pi_p\right).$$

Recall that $\mathbb{T}(U^p)$ is a compact topological ring with finitely many maximal ideals, so $\mathbb{T}(U^p) = \prod_{\mathfrak{m}} \mathbb{T}(U^p)_{\mathfrak{m}}$. Given $\mathfrak{m}$ we had a continuous semisimple $\overline{r}_{\mathfrak{m}} : G_F \to GL_n(k(\mathfrak{m}))$, which we called Eisenstein if it is absolutely reducible and non-Eisenstein if it is irreducible.

For now, assume $\overline{r}_{\mathfrak{m}}$ is non-Eisenstein. Then it extends to $r_{\mathfrak{m}} : G_{F^+} \to \mathscr{G}_n(\mathbb{T}(U^p)_{\mathfrak{m}})$.

Inside $\mathscr{A}^{cts}(G(F^+)\backslash G(\mathbb{A}^\infty_{F^+})/U^p, L)$, which is an admissible Banach rep of $G(F_p^+)$, we have $\mathscr{A}^{la}(G(F^+)\backslash G(\mathbb{A}^\infty_{F^+})/U^p, L)$, the functions which are locally on $G(F_p^+)$ analytic. This can also be obtained as the locally analytic vectors in $\mathscr{A}^{cts}$ as a rep of $G(F_p^+)$, and so is itself an admissible locally analytic representation of $G(F_p^+)$.

$\mathscr{A}^{la}(G(F^+)\backslash G(\mathbb{A}^\infty_{F^+})/U^p, L)$ has an action of the larger ring $\mathbb{T}(U^p)^{rig}$ (we are going to drop the U^p from now on) which can be defined abstractly as $\mathcal{O}((\operatorname{Spf}\mathbb{T})^{an})$; it is a Fréchet-Stein algebra. More concretely, $\mathbb{T}^{rig} = \prod_{\mathfrak{m}} \mathbb{T}^{rig}_{\mathfrak{m}}$, where

$$\mathbb{T}^0_{\mathfrak{m},n} = \left(\mathbb{T}_{\mathfrak{m}}\left[\frac{\mathfrak{m}^n}{p}\right]\right)^{\wedge}_p \subset \mathbb{T}_{\mathfrak{m}}[1/p]$$

and

$$\mathbb{T}^{rig}_{\mathfrak{m}} = \varprojlim_n \mathbb{T}^0_{\mathfrak{m},n}[1/p].$$

For example, if $\mathbb{T}$ were $\mathbb{Z}_p[\![T]\!]$, then $\mathbb{T}^0_{\mathfrak{m},n}$ would be the rigid functions on a closed disc of radius < 1, and $\mathbb{T}^{rig}$ would be the rigid functions on the open unit disc, so this would come out to the usual rigid generic fiber. We will write $X^{Hecke} = (\operatorname{Spf}\mathbb{T})^{an}$ when we want to think of it as a rigid space rather than a ring.

The image of the locally algebraic forms lies in the locally analytic space, so we get an embedding

$$\bigoplus_{\underline{k}} \mathscr{A}_{\underline{k}}(G(F^+)\backslash G(\mathbb{A}^\infty_{F^+})/U^p, L) \otimes \sigma_{\underline{k}} \hookrightarrow \mathscr{A}^{la}(G(F^+)\backslash G(\mathbb{A}^\infty_{F^+})/U^p, L).$$

The motivation for what we will do next is as follows. In the classical setting, only Eisenstein series move continuously; cusp forms are rigid. But p-adically, some cusp forms also move continuously. For both, this happens because they're inductions of characters that move continuously, at least locally at p. So we want to understand the passage between the induction of a character from a Borel and the representation of the full group $G(F_p^+)$—that is, we want to start with the full representation and pick out the varying character, which we do with the Jacquet module.

16.3 Emerton's Jacquet module

We're going to start with a more abstract perspective. Last year, we worked with admissible locally analytic representations, whose strong duals were coadmissible, or coherent sheaves on the Fréchet-Stein algebra of locally analytic distributions. For the Jacquet module we need something slightly more general, called "essential admissibility", which we didn't describe last year, so we'll do that now.

Let Z be a locally analytic abelian group such that $Z/$(some compact open subgroup) is finitely generated (e.g. $K^\times$, $T(K)$). Last year, we defined $D^{la}_c(Z, L)$, the dual of the space of locally analytic functions on Z with values in L, that is the locally analytic distributions on Z of compact support. This is a Fréchet-Stein algebra.

We can instead look at $\mathscr{O}(\mathscr{C}(Z))$, which is also a Fréchet-Stein algebra. We have a natural map

$$\begin{aligned} D_c^{la}(Z,L) &\to \mathscr{O}(\mathscr{C}(Z)) \\ \mu &\mapsto (x \mapsto \mu(x)). \end{aligned}$$

This is an isomorphism if Z is compact, but not in general, as in the following example.

Example 16.3.1.

$$D_c^{la}(\mathbb{Z},L) = \bigoplus_{n\in\mathbb{Z}} L\delta_n \cong L[T^{\pm 1}]$$

where δ_n is the delta function at n and corresponds to T^n. On the other hand

$$\mathscr{O}(\mathscr{C}(\mathbb{Z})) = \mathscr{O}(\mathbb{G}_m^{an}) = \left\{ \sum_{-\infty}^{\infty} a_n T^n \mid \forall A \in \mathbb{R}_{>1}, |a_n|A^n \to 0 \text{ and } |a_{-n}|A^n \to 0 \text{ as } n \to \infty \right\}$$

The map $D_c^{la}(Z,L) \to \mathscr{O}(\mathscr{C}(Z))$ is the natural embedding.

Now let H/K be reductive and V a locally analytic rep of $H(K)$ on a locally convex topological vector space of compact type. We saw last year that $D_c^{la}(H(K),L)$ acts on the strong dual V_b' and that V is admissible if and only if V_b' is coadmissible over $D_c^{la}(U,L)$ for some (hence any) open compact subgroup $U \subset H(K)$. An admissible V is sort of like a coherent sheaf over the space corresponding to $D_c^{la}(H(K),L)$ except that the latter isn't commutative.

Definition 16.3.2. We call V essentially admissible if the $D_c^{la}(Z(H)(K),L)$-action on V_b' extends to an action of $\mathscr{O}(\mathscr{C}(Z(H)(K)))$ and V_b' is coadmissible as a module over the bigger Fréchet-Stein algebra

$$D_c^{la}(U,L)\widehat{\otimes}_{D_c^{la}(Z(H)(K)\cap U,L)}\mathscr{O}(\mathscr{C}(Z(H)(K)))$$

for some (hence any) open compact subgroup $U \subset H(K)$. This is the "sensible" way to treat the noncompactness of the center.

So for example admissible representations of $\mathbb{Z}$ are finite-dimensional over L but essentially admissible representations, such as $\mathscr{O}(\mathscr{C}(Z))$ itself, are not. $\mathbb{G}_m$ has a copy of $\mathbb{Z}$ so behaves similarly.

Example 16.3.3. If T is a torus, there is an equivalence of categories between
{essentially admissible locally analytic reps of $T(K)$} and
{coherent sheaves on $\mathscr{C}(T(K))$}
taking V to V_b'. (This is by the definition of coadmissible.)

Suppose $H \supset P = MN$ where P is parabolic, M is the Levi, and N is the unipotent radical of P. Emerton's functor J_P takes
{(essentially) admissible locally analytic representations of $H(K)$} to
{(essentially) admissible locally analytic representations of $M(K)$}.

Usually Jacquet modules are given by coinvariants, but there's a different classical definition that uses invariants instead and generalizes better (the classical theory goes between the two definitions to get different properties). The resulting J_P is $V \mapsto (V^{N_0})_{fs}$, where $N_0 \subset N$ is any compact open subgroup and fs (finite-slope) means roughly the following. V^{N_0} has an action by the semigroup

$$M(K)^+ = \{g \in M(K) \mid gN_0g^{-1} \subset N_0\}$$

which is given by

$$g.v = \frac{1}{[N_0 : gN_0g^{-1}]} \sum_{n \in N_0/gN_0g^{-1}} ngv$$

(which makes sense since we know $gN_0g^{-1} \subset N_0$). Then the finite-slope part is the part where the $M(K)^+$-action extends to an $M(K)$-action (i.e. acts invertibly). The important thing about this is that the output is essentially admissible. Also it is left exact (the usual Jacquet module is actually exact, but for this Emerton has a counterexample). It commutes with the action of $Z(H)(K)$.

Given Lie(H) and $\mathfrak{U}(\mathrm{Lie}(H))$, the center $\mathfrak{Z}(\mathrm{Lie}(H))$ acts on V while $\mathfrak{Z}(\mathrm{Lie}(M))$ acts on $J_P(V)$ (since $M(K)$ acts on $(V^{N_0})_{fs}$). The Harish-Chandra map $\mathfrak{Z}(\mathrm{Lie}(H)) \to \mathfrak{Z}(\mathrm{Lie}(M))$, suitably normalized, is compatible with J_P.

For $B \subset G/F_p^+$ a Borel and $T \subset B$ its maximal torus, consider

$$J_B(\mathscr{A}^{la}(G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)/U^p, L))\left(\left(| \cdot |^{i-n} \prod_\tau \tau^{1-i}\right)\right) =: Jac_B(\mathscr{A}^{la}).$$

Here the LHS has an action of $T(F_p^+)$ and we twist it by the given character, where τ runs over the embeddings $F_p^+ \to \overline{L}$. This is an essentially admissible representation of T and produces a coherent sheaf $\mathscr{M}(U^p)$ on $\mathscr{C}_{F,n}^{csd}$. Both have an action of $\mathbb{T}(U^p)^{rig}$.

Next time, we'll do much more with this.

17 March 11: the eigenvariety.

17.1 More on Jacquet modules

Recall: let H be reductive over $K/\mathbb{Q}_p$, with parabolic subgroup $P = MN$. We defined J_P, Emerton's Jacquet functor taking

{essentially admissible locally analytic representations of $H(K)$} to

{essentially admissible locally analytic representations of $M(K)$},

by $J_P(V) = (V^{N_0})_{fs}$, where $N_0 \subset N(K)$ is an open compact subgroup, $M(K)^+ = \{m \in M(K) \mid mN_0m^{-1} \subset N_0\}$ acts on V^{N_0}, and $(V^{N_0})_{fs}$ is the largest subspace where this extends to an action of $M(K)$. This is left exact.

For $B = TN$ a Borel in G, so that $\mathscr{C}(T(F_p^+)) = \mathscr{C}_{n,F}^{csd}$, we defined

$$Jac_B(\mathscr{A}^{la}(G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)/U^p, L)) = J_B(\mathscr{A}^{la}(G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)/U^p, L))\left(\left(| \cdot |^{i-n} \prod_\tau \tau^{1-i}\right)_i\right).$$

This has an action of $\mathbb{T}(U^p)^{rig}$ and an analytic action of $T(F_p^+)$, and is essentially admissible as a representation of $T(F_p^+)$, hence corresponds to a coherent sheaf $\mathscr{M}_{U^p}$ on $\mathscr{C}_{n,F}^{csd}$. By picking out some element of $T(F_p^+)$ that isn't in the maximal compact subgroup (the characters of which are given by $\mathscr{W}_{n,F}$), we have a factorization

$$\mathscr{C}_{n,F}^{csd} \to \mathbb{G}_m^{an} \times \mathscr{W}_{n,F}^{csd} \to \mathscr{W}_{n,F}^{csd}$$

and it turns out $\mathscr{M}_{U^p}$ is already coherent over $\mathbb{G}_m^{an} \times \mathscr{W}_{n,F}^{csd}$. The theory of the Jacquet module tells us that the support of $\mathscr{M}_{U^p}$ is cut out in $\mathbb{G}_m^{an} \times \mathscr{W}_{n,F}^{csd}$ by a Fredholm series, that is, one everywhere convergent analytic function. From this one can conclude that

- each irreducible component of $\operatorname{supp} \mathscr{M}$ has Zariski open image in $\mathscr{W}_{n,F}^{csd}$ (so lives over the whole of weight space except possibly missing lower-dimensional pieces), and
- $\operatorname{supp} \mathscr{M}$ has an admissible cover $\{U_i\}$ such that for each i, there is a finite map $U_i \twoheadrightarrow W_i \subset \mathscr{W}_{n,F}^{csd}$ with W_i admissible open in $\mathscr{W}_{n,F}^{csd}$, and $\mathscr{M}|_{U_i}$ is finite projective over $\mathscr{O}_{W_i}$.

One should think of $\mathscr{M}_{U^p}$ as a sheaf of automorphic forms with nontrivial Jacquet module (or that are principal series at p). The fibers at a given point of weight space are modular forms where T is acting on the Jacquet module by the given character.

17.2 Construction of the eigenvariety

Definition 17.2.1. Let $\mathscr{H} \subset \underline{\operatorname{End}}_{\mathscr{O}_{\mathscr{C}_{F,n}^{csd}}}(\mathscr{M})$ be the $\mathscr{O}_{\mathscr{C}_{F,n}^{csd}}$-subalgebra generated by the image of $\mathbb{T}(U^p)^{rig}$. (Again we are taking the [trianguline part of the] Hecke algebra and "spreading it out" over the space of characters.)

Let $\mathscr{E} = \operatorname{Sp}(\mathscr{H})$. This is the eigenvariety.

We have a map

$$\mathscr{E} \to \operatorname{supp}(\mathscr{M}) \to \mathscr{W}_{F,n}^{csd}$$

such that $\mathscr{E} \to \mathscr{W}_{F,n}^{csd}$ is locally on the source finite (because $\mathscr{M}$ is finite projective over $\mathscr{O}_{\mathscr{W}_{F,n}^{csd}}$ and so $\mathscr{H}$ is a locally free sheaf). We furthermore have a map

$$\mathscr{E} \to X^{Hecke} = (\operatorname{Spf} \mathbb{T}(U^p))^{rig}$$

coming from the map $\mathbb{T}(U^p)^{rig} \to \mathscr{H}$ (remember that $\mathbb{T}(U^p)^{rig} = \mathscr{O}(X^{Hecke})$). Also keep in mind that

$$\operatorname{supp} \mathscr{M} \subset \mathscr{C}_{F,n}^{csd} \to \mathscr{W}_{F,n}^{csd}.$$

Here are some facts that come out of the properties of $\operatorname{supp}(\mathscr{M})$.

- Every irreducible component of $\mathscr{E}$ has Zariski open image in $\mathscr{W}_{F,n}^{csd}$.
- $\mathscr{E}$ has an admissible cover $\{U_i\}$ such that $U_i \to W_i \subset \mathscr{W}_{F,n}^{csd}$ where $U_i \twoheadrightarrow W_i$ is finite and $W_i \subset \mathscr{W}_{F,n}^{csd}$ is open admissible.
- $\mathscr{E}$ has no embedded components, meaning that if $\mathfrak{p}$ is a prime such that $\mathscr{O}_{\mathscr{E},x}^{\wedge}/\mathfrak{p} \hookrightarrow \mathscr{O}_{\mathscr{E},x}^{\wedge}$, then $\mathfrak{p}$ is minimal. (Intuitively, there is no "funny behavior" at individual points—all properties spread out throughout the eigenvariety.)

- $\mathscr{E} = \coprod_{\mathfrak{m} \subset \mathbb{T}(U^p)} \mathscr{E}_{\mathfrak{m}}$.
- $\mathscr{E} \hookrightarrow X^{Hecke} \times \mathscr{C}^{csd}_{F,n}$ (since $\mathscr{H}$ is generated by the Hecke operators and $\mathscr{O}_{\mathscr{C}^{csd}_{F,n}}$).

Suppose $(\theta, \underline{\delta}) \in X^{Hecke} \times \mathscr{C}^{csd}_{F,n}$ (so θ is a character of the Hecke algebra and $\underline{\delta}$ a character of the maximal torus). When does $(\theta, \underline{\delta}) \in \mathscr{E}$? By definition, this happens when

$$J_B(\mathscr{A}^{la}(G(F^+)\backslash G(\mathbb{A}^\infty_{F^+})/U^p, L)) \left[\theta, \underline{\delta}\left(|\cdot|^{n-i}\prod_\tau \tau^{i-1}\right)\right] \neq (0).$$

J_B only affects the representation of $G(F_p^+)$ and is left exact, so the $[\theta]$ (i.e. modding out by $\ker\theta$) can be moved inside it. So the above is equivalent to

$$J_B(\mathscr{A}^{la}(G(F^+)\backslash G(\mathbb{A}^\infty_{F^+})/U^p, L)[\theta]) \left[\underline{\delta}\left(|\cdot|^{n-i}\prod_\tau \tau^{i-1}\right)\right] \neq (0).$$

We know that this contains

$$\bigoplus_k J_B(\mathscr{A}_k(G(F^+)\backslash G(\mathbb{A}^\infty_{F^+})/U^p, L)[\theta] \otimes \sigma_k) \left[\underline{\delta}\left(|\cdot|^{n-i}\prod_\tau \tau^{i-1}\right)\right].$$

Now $\mathscr{A}_k(G(F^+)\backslash G(\mathbb{A}^\infty_{F^+})/U^p, L)$ is a smooth representation, so a sufficiently small N_0 acts trivially on it, and taking N_0-invariants of the stuff inside the J_B only affects σ_k, where it picks out the highest weight, i.e. the part where the torus is acting by k. So the above expression is (0) unless

$$wt_\tau(\delta_i) = -k_{\tau,i} + i - 1.$$

In this case, we get

$$J_B(\mathscr{A}_{(i-1-wt(\delta_i))}(G(F^+)\backslash G(\mathbb{A}^\infty_{F^+})/U^p, L)[\theta]) \left[\underline{\delta}\left(|\cdot|^{n-i}\prod_\tau \tau^{wt_\tau(\delta_i)}\right)\right]$$

(where this J_B is now just the "usual" Jacquet module). Tensoring this by $\otimes_{L,\tau}\mathbb{C}$, we get

$$\bigoplus_{\pi:\pi_\infty \cong \sigma_{(i-1-wt(\delta_i))}} (\pi^{p,\infty})^{U^p}[\theta] \otimes J_B(\pi_p) \left[\underline{\delta}\left(|\cdot|^{n-i}\prod_\tau \tau^{wt_\tau(\delta_i)}\right)\right].$$

Elements of this space arise from "refined automorphic representations"

$$RA_{n,U^p} = \{(\pi, \underline{\chi}) \mid \pi \text{ is an automorphic rep of } G(\mathbb{A}), \mathbb{C}(\underline{\chi}) \hookrightarrow J_B(\pi_p)(|\cdot|^{i-n})\}/\sim$$

where $(\pi, \underline{\chi}) \sim (\pi', \underline{\chi}')$ if $\underline{\chi} = \underline{\chi}'$ and $\pi^S \cong (\pi')^S$. (By strong multiplicity one these equivalence classes would be singletons for GL_n, but maybe not for U_n. The construction of the eigenvariety as we wrote it just records what's happening at p and the good places, and doesn't track the other bad places. You could add that, but we won't need it.)

Fixing $\mathbb{C} \cong \overline{\mathbb{Q}}_l$, we have

$$\begin{aligned} RA_{n,U^p} &\hookrightarrow \mathscr{E} \subset X^{Hecke} \times \mathscr{C}^{csd}_{F,n} \\ (\pi, \underline{\chi}) &\mapsto \left(\theta_\pi, \left(\chi_i \prod \tau^{k_{\tau,i}+1-i}\right)\right) \end{aligned}$$

where $\theta_\pi : \mathscr{H}_0^S \to \mathbb{C}$ is the character such that $\pi^{S,U^S} \cong \mathbb{C}[\theta_\pi]$ (note that U^S is hyperspecial maximal compact and π^{S,U^S} must be 1-dimensional) and $\pi_\infty \cong \sigma_k$. Note that each π gives a unique θ_π, but it may give rise to more than one refinement $\underline{\chi}$.

Lemma 17.2.2. *$\mathbb{C}(\underline{\chi}) \hookrightarrow J_B(\pi_p)(|\cdot|^{i-n})$ if and only if for all $v|p$,*

$$\mathrm{rec}\left(\pi_v|\cdot|^{\frac{1-n}{2}}\right) = WD(r_p(\pi)|_{G_{F_v}})^{F-ss}$$

has a triangulation parameter $\underline{\chi}$. Equivalently, for all $v|p$, $WD(r_p(\pi)|_{G_{F_v}})$ has a triangulation parameter $\underline{\chi}$.

To prove this, you look at the classification of irreducible smooth representations of $GL_n(F_v)$. The supercuspidal support of J_B must be a bunch of characters, which is the same as WD^{F-ss} being a direct sum of characters, and if you calculate them the characters match up. Furthermore, a WD rep has a triangulation with a given parameter if and only if its Frobenius semisimplification does (this is a quick calculation from the definitions).

Corollary 17.2.3. *If $(\pi, \underline{\chi}) \in RA_{n,U^p}$, then for all $v|p$, $\mathscr{M}_{rig}(r_p(\pi)|_{G_{F_v}})$ is trianguline.*

The triangulation character can't necessarily be determined, but it will be with further conditions. We call $(\pi, \underline{\chi}) \in RA_{n,U^p}$

- regular if $\underline{\chi}$ is regular, and
- numerically noncritical if for all τ, i,

  $$\max\left\{0, v_p((\chi_1 \cdots \chi_i)(p)) + \sum_\tau \sum_{j \le i} k_{\tau,j} + \frac{1}{2} i(i-1)[K:\mathbb{Q}_p]\right\} < 1 + k_{\tau,i} - k_{\tau,i+1}$$

  where $k_{\tau,j}$ are the weights of π_∞.

Corollary 17.2.4. *If $(\pi, \underline{\chi}) \in RA_{n,U^p}$ is numerically noncritical, then $\mathscr{M}_{rig}(r_p(\pi)|_{G_{F_v}})$ is trianguline with parameter $\left(\chi_i \prod \tau^{k_{\tau,i}+1-i}\right)$ for all $v|p$ (and this is numerically noncritical). (Otherwise, the $\prod \tau^{k_{\tau,i}+1-i}$s might be permuted among the triangulation factors.)*

17.3 Geometry of the eigenvariety

Recall that if a point on the trianguline variety is numerically noncritical or generic, then it is trianguline with the "expected" parameter. The same is true for the eigenvariety, as follows.

Proposition 17.3.1 (Classicality criterion)**.** *1. (Emerton) If $(x, \underline{\delta}) \in \mathscr{E}$ and $\underline{\delta}$ is numerically noncritical, then $(x, \underline{\delta})$ is in the image of RA_{n,U^p}.*

2. *(Newton-Thorne after Breuil) If $(x, \underline{\delta}) \in \mathscr{E}$ and $r_x|_{G_F}$ is irreducible (Richard isn't sure if this is necessary) and generic for all $v|p$, and $\underline{\delta}$ is regular, then $(x, \underline{\delta})$ is in the image of RA_{n,U^p}.*

The proof of this is analytic/representation-theoretic rather than Galois-theoretic, but it mirrors the Galois argument for the trianguline variety.

Suppose $(x, \underline{\delta}) \in \mathscr{E}$ and $wt_\tau(\delta_i) \in \mathbb{Z}$ (this could even be $\mathbb{Z}_p$) for all τ, i. Let U be an affinoid neighborhood of $(x, \underline{\delta})$ that is an admissible open of $\mathscr{E}$ with a finite surjective map onto the affinoid $W \subset \mathscr{W}^{csd}_{F,n}$. Since affinoids behave as if they are compact, we have $v_p(\delta_i(p)) < C$ over U for some constant $C > 0$. Consider

$$\underline{\delta}' = \underline{\delta}|_{(\mathscr{O}^\times_{F_p})^n} \prod_\tau (\tau| \cdot |_p)^{-m_{\tau,i}}$$

where $m_{\tau,i} \in \mathbb{Z}$. Suppose this is very close to $\underline{\delta}$ in $\mathscr{W}^{csd}_{F,n}$. In particular, we should have $\#(\mathscr{O}^\times_{F,p})^{tor} | m_{\tau,i}$, and $m_{\tau,i}$ should be sufficiently close to 0 p-adically. Furthermore we should include an archimedean condition

$$m_{\tau,i+1} > nC + wt_\tau(\delta_i) - wt_\tau(\delta_{i+1}) + m_{\tau,i}.$$

In this case $\underline{\delta}' \in W$. That is, we can find a set of $(x', \underline{\delta}') \in U$ of this form tending to $(x, \underline{\delta})$ which is Zariski dense in U, such that $(\underline{x}', \underline{\delta}')$ is numerically noncritical (that's the archimedean condition). We conclude that any point $(x, \underline{\delta}) \in \mathscr{E}$ with $wt_\tau(\delta_i) \in \mathbb{Z}$ (or $\mathbb{Z}_p$) for all τ, i is the p-adic limit of a sequence of numerically noncritical elements of RA_{n,U^p}, which are also locally Zariski dense.

If $\underline{\delta}$ is regular, we can choose the approximating points to be regular too. Consequently, regular, numerically noncritical points in RA_{n,U^p} are Zariski dense in any irreducible component of $\mathscr{E}$ (since the image of that component is Zariski open in weight space). We can even ensure that these points are n-regular for any $n \in \mathbb{Z}_{>0}$, meaning that $(\chi_i/\chi_j)^k \neq 1$ for $i \neq j$ and $1 \leq k < n$. This will come up because we will need a representation of GL_2 to be n-regular for its symmetric power to be regular.

The overall point is that by Emerton's classicality criterion, there are lots of classical points. They aren't p-adically dense everywhere, but they have many accumulation points.

As a consequence, $\mathscr{E}$ is reduced. This is because "no embedded components" means that it can't be nonreduced at a single point; if it's nonreduced it has to be nonreduced along a whole irreducible component, hence at a classical point, but that would be equivalent to the classical Hecke algebra being nonreduced, and in fact the classical Hecke algebra generated by the Hecke operators at good primes is reduced. (If you included operators at bad primes you might get nonreducedness.)

Furthermore, if $\mathfrak{m}$ is non-Eisenstein (so $\overline{r}_\mathfrak{m}$ is absolutely irreducible), then we have the global trianguline variety $\mathscr{T}_{\overline{r}_m}$ parametrizing trianguline deformations of $\overline{r}_\mathfrak{m}$ with a triangulation etc. We have

$$\mathscr{T}_{\overline{r}_m} \subset X^{univ}_{\overline{r}_\mathfrak{m}} \times \mathscr{C}^{csd}_{F,n} \supset X^{Hecke}_\mathfrak{m} \times \mathscr{C}^{csd}_{F,n}$$

where the containment on the right comes from the universal representation $r^{univ}_\mathfrak{m} : G_{F^+} \to \mathscr{G}_n(\mathbb{T}_\mathfrak{m})$, which gives a surjection $R^{univ}_{\overline{r}_\mathfrak{m},S} \twoheadrightarrow \mathbb{T}_\mathfrak{m}$ (so $X^{Hecke}_\mathfrak{m} \subset X^{univ}_{\overline{r}_\mathfrak{m}}$ is a closed embedding). We know that

$$\mathscr{E}_\mathfrak{m} \subset X^{Hecke}_\mathfrak{m} \times \mathscr{C}^{csd}_{F,n}$$

and in fact

$$\mathscr{E}_{\mathfrak{m}} \subset \mathscr{T}_{\bar{r}_{\mathfrak{m}}}$$

because regular numerically noncritical points of RA_{n,U^p} are Zariski dense in $\mathscr{E}_{\mathfrak{m}}$, and these lie in $\mathscr{T}_{\bar{r}_{\mathfrak{m}}}$ because the Galois representation is trianguline with the given parameter. The natural conjecture is that $\mathscr{E}_{\mathfrak{m}}$, the trianguline automorphic things, actually equals $\mathscr{T}_{\bar{r}_{\mathfrak{m}}}$, the trianguline Galois things. We can frequently prove something like this by showing that the two have the same tangent space dimension (i.e. the dimension of weight space) at a nice classical point, so that any component of $\mathscr{T}_{\bar{r}_{\mathfrak{m}}}$ containing a classical point must equal a component of $\mathscr{E}_{\mathfrak{m}}$ (but technically there could be a component that contains no classical points). More precisely:

Theorem 17.3.2. *Suppose $\mathfrak{m}$ is non-Eisenstein and that $\mathscr{C}$ is an irreducible component of $\mathscr{T}_{\bar{r}_{\mathfrak{m}}}$ containing a regular noncritical point $(\pi, \underline{\chi}) \in RA_{n,U^p}$ such that $r_p(\pi)(G_{F(\zeta_{p^\infty})})$ is "enormous" (has an element with distinct eigenvalues etc.). Then $\mathscr{C} \subset \mathscr{E}_{\mathfrak{m}}$. In particular, if $(x, \underline{\delta}) \in \mathscr{C}$ with either*

1. *$\underline{\delta}$ numerically noncritical or*

2. *$r_x|_{G_{F_v}}$ irreducible and generic for all $v|p$ and $\underline{\delta}$ regular,*

then $(x, \underline{\delta})$ is in the image of RA_{n,U^p}. In particular r_x is automorphic.

(The "enormous" condition won't be a serious obstruction: if the thing you're taking a symmetric power of has open image in GL_2, which is usually the case if you have a compatible system of l-adic representations that are not CM, you'll get it by a maybe slightly generalized theorem of Serre.)

Proof. As discussed above, we have

$$\dim T_{(\pi,\underline{\chi})}\mathscr{T}_{\bar{r}_{\mathfrak{m}}} \leq \dim \mathscr{W}^{csd}_{F,n} = \dim(\text{every component of } \mathscr{E}).$$

$\square$

We are currently assuming that $\mathfrak{m}$ is non-Eisenstein, but we will crucially need to deal with the case where $\mathfrak{m}$ is Eisenstein, so we'll do that next. It's not too much harder as long as you're in characteristic 0. (The reason we need this is that mod p, as you take symmetric powers, you very quickly become reducible. This is good for the part of their argument producing one lift that we won't talk about, because it's easier to prove things are automorphic if they're reducible, but an annoyance here.) Then we'll talk about automorphy of symmetric powers.

18 March 18: Eisenstein ideals and symmetric powers.

18.1 Pseudo-representations

Recall the following theorem from last week.

Theorem 18.1.1. *Let $\mathfrak{m}$ be a non-Eisenstein maximal ideal of $\mathbb{T}$, $\overline{r}_{\mathfrak{m}}$ its associated Galois representation, and $\mathscr{C}$ an irreducible component of $\mathscr{T}_{\overline{r}_{\mathfrak{m}}}$ containing a regular, noncritical point (π, χ) coming from RA_{n,U^p} with $r_p(\pi)(G_{F(\zeta_{p^\infty})})$ enormous. Then $\mathscr{C} \subset \mathscr{E}_{\mathfrak{m}}$. In particular, if $(x, \underline{\delta}) \in \mathscr{C}$ with either*

1. *$\underline{\delta}$ numerically noncritical or*

2. *$r_x|_{G_{F_v}}$ irreducible and generic for all $v|p$ and $\underline{\delta}$ regular,*

then $(x, \underline{\delta})$ is in the image of RA_{n,U^p} and r_x is automorphic.

We need to get rid of the non-Eisenstein assumption, because if we take high symmetric powers of two-dimensional representations, once the power exceeds the residue characteristic, we never get non-Eisenstein points. Where did this assumption come from? We used the map $R^{univ}_{\overline{r}_{\mathfrak{m}}} \twoheadrightarrow \mathbb{T}_{\mathfrak{m}}$, which does not exist in the non-Eisenstein case; there might be some replacement for $R^{univ}_{\overline{r}_{\mathfrak{m}}}$, but the more serious problem is that $r^{auto}_{\mathfrak{m}} : G_F \to GL_n(\mathbb{T}_{\mathfrak{m}})$ does not exist if $\overline{r}_{\mathfrak{m}}$ is reducible. So instead we use pseudo-representations, which capture what's important about a representation but are slightly weaker.

Let Γ be a group, R a ring, and $T : \Gamma \to R$ a class function, meaning $T(gh) = T(hg)$ for all $g, h \in \Gamma$. Given elements $g_1, \ldots, g_n \in \Gamma$ and $c = (i_1, \ldots, i_r) = (i_r, i_1, \ldots, i_{r-1}) = \cdots$ a cycle in S_n (i.e. $i_1, \ldots, i_r$ is a list of elements of $\{1, \ldots, n\}$ without repetition), let

$$T_c(g_1, \ldots, g_n) = T(g_{i_1} \cdots g_{i_r}).$$

This is well-defined because T is a class function, so you can move the g_{i_r} to the left.

In fact, if $\sigma \in S_n$, we can write $\sigma = c_1 \cdots c_s$ as a product of disjoint cycles (uniquely up to order), and define

$$T_\sigma(g_1, \ldots, g_n) = T_{c_1}(g_1 \ldots, g_n) T_{c_2}(g_1, \ldots, g_n) \cdots T_{c_s}(g_1, \ldots, g_n).$$

This is well-defined because $c_1 \cdots c_s$ is unique up to order and multiplication in R is commutative.

Definition 18.1.2. By a pseudo-representation $T : \Gamma \to R$ of dimension d, we mean a class function $T : \Gamma \to R$ such that $T(1) = d$ and

$$\sum_{\sigma \in S_{d+1}} (-1)^\sigma T_\sigma(g_1, \ldots, g_{d+1}) = 0$$

for all $g_1, \ldots, g_{d+1} \in \Gamma$ (where by $(-1)^\sigma$ we mean $\operatorname{sign}(\sigma)$).

Proposition 18.1.3.

1. *If $r : \Gamma \to GL_n(R)$ is a representation, then $\operatorname{tr}(r)$ is a pseudo-representation of dimension d.*

2. *If R is an algebraically closed field of characteristic 0, and $T : \Gamma \to R$ is a pseudo-representation of dimension d, then there is a semisimple representation $r : \Gamma \to GL_d(R)$ with $\operatorname{tr}(r) = T$ (note that r can be chosen to be semisimple because the semisimplification of a representation has the same trace). Moreover, r is unique up to conjugation.*

Proof. The proof of Part 1 is by the Cayley-Hamilton theorem—if you take the resulting polynomial relation and write it out as a multi-linear relation, that's the equation you get.

The point of Part 2 is that it's a standard theorem that representations in characteristic 0 are determined by their trace. So you can ask what properties does a function have to have in order to be the trace of a true representation, and the given relations capture that. □

Remark 7.

1. In small nonzero characteristics, when a representation is not necessarily determined by its trace, there are "better" notions than "pseudo-representations" that involve $\operatorname{tr} \wedge^i r$ for $i = 1, \ldots, d$ (since a representation is still determined by these, or equivalently by the coefficients of the characteristic polynomials). Unfortunately, the equations needed are not explicit. You can use

   - "determinants" by Chenevier, or
   - "pseudo-characters" by Vincent Lafforgue,

   and they are actually equivalent. Newton-Thorne use these, but for our discussion we won't need to.

2. If Γ and R have topologies, we can talk about continuous pseudo-representations, and the previous theorem remains true with T and r assumed continuous. (That is, if T is continuous, the corresponding r can be chosen to be continuous.)

3. Pseudo-representations of dimension 1 are characters (the relation becomes $T(g_1g_2) = T(g_1)T(g_2)$).

If $\chi : \Gamma \to R^\times$ is a character and $T : \Gamma \to R$ a pseudo-representation of dimension d,

$$(T \otimes \chi)(g) = T(g)\chi(g)$$

is a pseudo-representation of dimension d such that $(\operatorname{tr} r) \otimes \chi = \operatorname{tr}(r \otimes \chi)$.

For $i = 0, \ldots, d$, if $i!$ is invertible in R, we can define

$$(\wedge^i T)(g) = \frac{1}{i!} \sum_{\sigma \in S_i} (-1)^\sigma T_\sigma(g, \ldots, g).$$

Then $\wedge^i T$ is a pseudo-representation of dimension $\binom{d}{i}$. In particular $\wedge^d T$ is a character. We have $\operatorname{tr}(\wedge^i r) = \wedge^i(\operatorname{tr} r)$. We can further define

$$T^\vee(g) = (\wedge^{n-1}T) \otimes (\wedge^n T)^{-1}$$

and we have $\operatorname{tr}(r^\vee) = (\operatorname{tr} r)^\vee$.

If $\rho \in \operatorname{Aut}(\Gamma)$, we can define $T^\rho(g) = T(\rho(g))$, and we have $\operatorname{tr}(r^\rho) = (\operatorname{tr} r)^\rho$.

We are going to be interested in conjugate self-dual representations r, which satisfy an equation of the form $r^\rho = r^\vee \otimes \chi$, which corresponds to

$$T^\rho(g) \sum_{\sigma \in S_d} (-1)^\sigma T_\sigma(g, \ldots, g) = d\chi(g) \sum_{\sigma \in S_{d-1}} (-1)^\sigma T_\sigma(g, \ldots, g).$$

In particular, if r is semisimple and $d!$ is invertible in R, $r^\rho \cong r^\vee \otimes \chi$ if and only if $\operatorname{tr}(r)$ satisfies the above equation.

18.2 Deformations of pseudo-representations

Let $L/\mathbb{Q}_p$ be finite, $\mathcal{O} = \mathcal{O}_L$, $\mathcal{O}/\lambda = \mathbb{F}/\mathbb{F}_p$ finite. Let Γ be a profinite group, and assume that for all open (hence finite-index) subgroups $\Delta \subset \Gamma$, $\dim_{\mathbb{F}_p} \operatorname{Hom}(\Delta, \mathbb{F}_p) < \infty$. (This is true if $\Gamma = G_K$ for K a local field, since a local field has only finitely many extensions of a given degree, or if $\Gamma = G_{F,S}$ for F a number field, S a finite set of places of F, F_S/F the maximal extension unramified outside S, and $G_{F,S} = \operatorname{Gal}(F_S/F)$.)

Let $\overline{T} : \Gamma \to \mathbb{F}$ be a continuous pseudo-representation of dimension d. A theorem of Nyssen says that there is

$$T^{univ} : \Gamma \to R_{\overline{T}}^{univ}$$

where $R_{\overline{T}}^{univ}$ is a complete noetherian local $\mathcal{O}$-algebra of residue field $\mathbb{F}$, and T^{univ} is a continuous pseudo-representation of dimension d such that $T^{univ} \pmod{\mathfrak{m}} = \overline{T}$ for $\mathfrak{m}$ the maximal ideal of $R_{\overline{T}}^{univ}$, and if $T : \Gamma \to R$ is any continuous pseudo-representation of dimension d deforming $\overline{T}$, then there is a unique $R_{\overline{T}}^{univ} \to R$ such that T^{univ} pushes forward to T.

(Note: this is "almost trivial". For true representations proving the existence of the universal representation is hard because two conjugate representations are considered the same, but here you just adjoin a variable for $T^{univ}(g)$ for each $g \in \Gamma$ and write down all the relations. The only hard part is showing that the result is noetherian, for which you need to use the technical hypothesis about the open subgroups of Γ.)

Also, there is a universal deformation satisfying $T^{\rho} = T^{\vee} \otimes \chi$ if $\chi : \Gamma \to \mathcal{O}^{\times}$ is a fixed character. Technically this isn't well-defined if p is smaller than the dimension of the pseudo-representation, but we really mean

$$T^{\rho}(g) \sum_{\sigma \in S_d} (-1)^{\sigma} T_{\sigma}(g, \dots, g) = d\chi(g) \sum_{\sigma \in S_{d-1}} (-1)^{\sigma} \operatorname{tr}(g, \dots, g)$$

for all $g \in \Gamma$, which is the same thing in characteristic 0.

Also by Nyssen, if $\overline{T} = \operatorname{tr} \overline{r}$ and $\overline{r}$ is absolutely irreducible, then the natural map $R_{\operatorname{tr} \overline{r}}^{univ} \to R_{\overline{r}}^{univ}$ is an isomorphism.

If $\mathfrak{m} \subset \mathbb{T}$ is a maximal ideal, Eisenstein or not, we have a continuous pseudo-representation

$$T^{auto} : G_{F,S} \to \mathbb{T}_{\mathfrak{m}}$$

of dimension n deforming $\operatorname{tr} \overline{r}_{\mathfrak{m}}$. Therefore, we have a canonical map

$$R_{\operatorname{tr} \overline{r}_{\mathfrak{m}}, S}^{univ} \to \mathbb{T}_{\mathfrak{m}}$$

(where $R_{\operatorname{tr} \overline{r}_{\mathfrak{m}}, S}^{univ}$ is with the conjugate self-dual condition), which is surjective after inverting p because $\mathbb{T}_{\mathfrak{m}}$ is generated by Hecke operators at good primes, which usually correspond to the trace of the corresponding Frobenius. (You have to invert p because actually the Hecke algebra is generated not only by traces of Frobenius elements but also other coefficients of the characteristic polynomials of Frobenius elements, which may have denominators when you express them in terms of the trace. Note that if you work with determinants/pseudocharacters the map is already surjective and you don't have to invert p.)

So as before we can work with the characteristic 0 fiber of the universal deformation

$$X_{\operatorname{tr} \overline{r}_{\mathfrak{m}}} = (\operatorname{Spf} R_{\operatorname{tr} \overline{r}_{\mathfrak{m}}, S}^{univ})^{an}.$$

If $x \in X_{\mathrm{tr}\,\overline{r}_{\mathfrak{m}}}$, we get a continuous semisimple true representation

$$r_x : G_{F,S} \to GL_n(\overline{k(x)}).$$

Similarly, defining $X_{\mathfrak{m}}^{Hecke} = (\operatorname{Spf} \mathbb{T}_{\mathfrak{m}})^{an}$ as before, we get an embedding

$$X_{\mathfrak{m}}^{Hecke} \hookrightarrow X_{\mathrm{tr}\,\overline{r}_{\mathfrak{m}}}.$$

Even though the residual representation may be reducible, we're mostly interested in irreducible deformations. Inside $X_{\mathrm{tr}\,\overline{r}_{\mathfrak{m}}}$ we have the irreducible locus

$$X_{\mathrm{tr}\,\overline{r}_{\mathfrak{m}}}^{irred} = \{x \text{ such that } r_x \text{ is irreducible}\}$$

which is a Zariski open subset of $X_{\mathrm{tr}\,\overline{r}_{\mathfrak{m}}}$, whose complement is defined by polynomial equations (see Chenevier's paper on determinants [4] for a proof—it doesn't matter whether you're using determinants or pseudo-representations, because once we're in the generic fiber, we're in characteristic 0 and they're equivalent). Locally on $X_{\mathrm{tr}\,\overline{r}_{\mathfrak{m}}}^{irred}$, one can globalize the representation: you get a map not necessarily into GL_n of the structure sheaf, but the multiplicative group of an Azumaya algebra over the structure sheaf, so that possibly after extending scalars to L'/L finite, there is

$$\text{`}r^{univ}\text{'} : G_{F,S} \to GL_n(\mathscr{O}_{X_{\mathrm{tr}\,\overline{r}_{\mathfrak{m}}}^{irred} \times L'}).$$

(Outside the irreducible locus this might not be true.) Again in $X_{\mathrm{tr}\,\overline{r}_{\mathfrak{m}}}^{irred} \times \mathscr{C}_{F,n}^{csd}$ we have the subset $\mathscr{T}_{\mathrm{tr}\,\overline{r}_{\mathfrak{m}}}^{reg}$ of points $(x, \underline{\delta})$ where $\underline{\delta}$ is regular and r_x is trianguline of parameter $\underline{\delta}$, whose Zariski closure is $\mathscr{T}_{\mathrm{tr}\,\overline{r}_{\mathfrak{m}}}$. (So $\mathscr{T}_{\mathrm{tr}\,\overline{r}_{\mathfrak{m}}}^{reg} \subset \mathscr{T}_{\mathrm{tr}\,\overline{r}_{\mathfrak{m}}}$ is Zariski dense, and $\mathscr{T}_{\mathrm{tr}\,\overline{r}_{\mathfrak{m}}} \subset X_{\mathrm{tr}\,\overline{r}_{\mathfrak{m}}}^{irred} \times \mathscr{C}_{F,n}^{csd}$ is Zariski closed. In fact $\mathscr{T}_{\mathrm{tr}\,\overline{r}_{\mathfrak{m}}}^{reg}$ is Zariski open in $\mathscr{T}_{\mathrm{tr}\,\overline{r}_{\mathfrak{m}}}$ by Kedlaya-Pottharst-Xiao [6].) We also have

$$\mathscr{E}_{\mathfrak{m}} \subset X_{\mathfrak{m}}^{Hecke,irred} \times \mathscr{C}_{F,n}^{csd} \subset X_{\mathfrak{m}}^{irred} \times \mathscr{C}_{F,n}^{csd}$$

with $\mathscr{E}_{\mathfrak{m}} \subset \mathscr{T}_{\mathrm{tr}\,\overline{r}_{\mathfrak{m}}}$ as before. Again we have the theorem from the start of lecture.

18.3 Symmetric power liftings

Let $\underline{\chi} = (\chi_1, \ldots, \chi_n)$ be a smooth character of $(F_p^{\times})^n$. We call $\underline{\chi}$ regular if $(\chi_i/\chi_j) \neq 1$ for $i \neq j$, and n-regular if $(\chi_i/\chi_j)^m \neq 1$ for $i \neq j$ and $1 \leq m \leq n-1$.

Let $\mathscr{E}_2$ be the eigenvariety for G_2, the rank 2 unitary group. Let $\mathscr{T}_n$ be the trianguline variety for GL_n. Consider the map

$$\begin{aligned} \operatorname{Sym}^{n-1} &: \mathscr{E}_2 \to \mathscr{T}_n \\ (x, (\delta_1, \delta_2)) &\mapsto (\operatorname{Sym}^{n-1} r_x, (\delta_1^{n-1}, \delta_1^{n-2}\delta_2, \ldots, \delta_2^{n-1})). \end{aligned}$$

A priori the image of this is only in $X^{univ} \times \mathscr{C}_{F,n}^{csd}$, and we need to check that it really lies in $\mathscr{T}_n$. But if x is n-regular and numerically noncritical, then r_x is trianguline with parameter (δ_1, δ_2), which implies that $\operatorname{Sym}^{n-1} r_x$ is trianguline with parameter $(\delta_1^{n-1}, \ldots, \delta_2^{n-1})$ and this is regular. Since the set of such x is Zariski dense, the map is as we said.

Let $(\pi, \underline{\chi}) \in RA_2$. Assume that

- $\underline{\chi}$ is n-regular and numerically noncritical,
- for any $v|p$, the Zariski closure $r_p(\pi)(G_{F,v})^{ZC} \supset SL_2$, and
- $\operatorname{Sym}^{n-1} r_p(\pi)$ is automorphic, and equals $r_p(\Pi)$ where Π is an automorphic representation of G_n.

Let $\mathscr{C}$ be the component of $\mathscr{E}_2$ containing $(\pi, \underline{\chi})$.

We claim that $\operatorname{Sym}^{n-1}\mathscr{C}$, which we know is contained in $\mathscr{T}_n$, is actually contained in $\mathscr{E}_n$. We know that $\operatorname{Sym}^{n-1}\mathscr{C}$ is connected, and it contains $(\Pi, (\chi_1^{n-1}, \ldots, \chi_2^{n-1})) \in RA_n$. According to the big theorem we stated, we need to check that

- $(\chi_1^{n-1}, \ldots, \chi_2^{n-1})$ is regular, which is true because $\underline{\chi}$ is n-regular.
- $(\operatorname{Sym}^{n-1} r_p(\pi))(G_{F(\zeta_{p^\infty})})$ is enormous.
- $(\operatorname{Sym}^{n-1} r_p(\pi), (\delta_1^{n-1}, \ldots, \delta_2^{n-1}))$ is noncritical. Keep in mind that we know

  $$\delta_1 = \chi_1 \prod_\tau \tau^{-k_{\tau,1}}, \qquad \delta_2 = \chi_2 \prod_2 \tau^{-k_{\tau,2}}$$

  where $\{k_{\tau,1} < k_{\tau,2}\} = HT_\tau(r_p(\pi))$, because we know $(r_p(\pi), (\delta_1, \delta_2))$ is noncritical.

To check noncriticality: we need

$$wt_\tau(\delta_1^{n-1}) < wt_\tau(\delta_1^{n-1}\delta_2) < \cdots < wt_\tau(\delta_2^{n-1})$$

or

$$(n-1)k_{\tau,1} < (n-2)k_{\tau,1} + k_{\tau,2} < \cdots < (n-1)k_{\tau,2}$$

which is true.

To check enormousness: if $r : \Gamma \to GL(V)$, let $\operatorname{ad} r$ be the representation with underlying vector space $\operatorname{End}(V)$ and $(\operatorname{ad} r)(g)(f) = r(g) \circ f \circ r(g)^{-1}$. Look at $\operatorname{ad}\operatorname{Sym}^{n-1} r_p(\pi)|_{G_{F(\zeta_{p^\infty})}}$. Since we assumed that the Zariski closure of the image of $r_p(\pi)$ contained SL_2, let std be the standard representation of SL_2. Then

$$\operatorname{ad}(\operatorname{Sym}^{n-1} std) = \operatorname{Sym}^{2n-2} std \oplus \operatorname{Sym}^{2n-4} std \oplus \cdots \oplus \operatorname{Sym}^0 std.$$

The irreducible components of a representation are the same as the irreducible components of the Zariski closure, which contains SL_2, so the irreducible components of

$$\operatorname{ad}\operatorname{Sym}^{n-1} r_p(\pi)|_{G_{F(\zeta_{p^\infty})}}$$

will be direct sums of the constituents of the above decomposition of $\operatorname{ad}(\operatorname{Sym}^{n-1} std)$. (Note that if the image of G_F contains SL_2, the same is true for $G_{F(\zeta_{p^\infty})}$, because $F(\zeta_{p^\infty})$ is an abelian extension of F, and SL_2 has no nontrivial abelian quotients.)

It suffices to find, for each constituent $\operatorname{Sym}^{2i} std$, an element $\sigma \in G_{F(\zeta_{p^\infty})}$ and an eigenvalue α of $\operatorname{Sym}^{n-1} r_p(\pi)(\sigma)$ such that

1. $\mathrm{Sym}^{n-1} r_p(\pi)(\sigma)$ has distinct eigenvalues (its characteristic polynomial has distinct roots), and

2. if $e_{\sigma,\alpha} \in \mathrm{End}(\mathrm{Sym}^{n-1} r_p(\pi))$ is the σ-equivariant projection to the α-eigenline of σ, then
$$\mathrm{tr}(e_{\sigma,\alpha} \mathrm{Sym}^{2i} std) \neq 0.$$

Again since the image of $r_p(\pi)$ is Zariski dense in SL_2, it suffices to find $\sigma \in SL_2$ with these properties. Let $\sigma = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$. Its eigenvalues are $t^{n-1}, t^{n-3}, \ldots, t^{1-n}$. These are all distinct if t is not a root of unity of small order. The eigenlines are spanned by $e_1^j e_2^{n-1-j}$ where e_1, e_2 is the standard basis of *std*. We want to find $A \in \mathrm{Sym}^{2i} std \subset \mathrm{ad}(\mathrm{Sym}^{n-1} std)$ with $\mathrm{tr}((\text{projection to } e_1^j e_2^{n-1-j})A) \neq 0$, i.e. $A_{jj} \neq 0$, for some j. That is, we want to know that each $\mathrm{Sym}^{2i} std \subset \mathrm{ad}(\mathrm{Sym}^{n-1} std)$ contains a matrix A with some nonzero diagonal entry. But if we think of $\mathrm{ad}(\mathrm{Sym}^{n-1} std)$ as a representation of $\left\{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right\}$ (with the action of conjugation by $\mathrm{Sym}\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$), the diagonal entries are just the 0-weight spaces. So we want to know that $\mathrm{Sym}^{2i} std$ has a 0-weight space for $\left\{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right\}$. This is in fact true for any even symmetric power of *std*.

So now we know that any component of $\mathscr{E}_2$ that contains a point where the local Galois representation has image containing SL_2, which is n-regular and numerically non-critical, and which has a symmetric power lifting, has symmetric power contained in the eigenvariety for GL_n. Next we need to deduce that the other points are also automorphic.

19 March 23: propagating symmetric power lifts.

19.1 Automatic genericity

Last time, we were looking at $\mathscr{E}_2$, the eigenvariety for G_2, the 2-variable unitary group over some totally real field. We picked some irreducible component $\mathscr{C} \subset \mathscr{E}_2$ containing a point $(\pi, \chi) \in RA_2$ (i.e. π is a cuspidal automorphic representation of G_2 and χ is a character of the maximal torus appearing in the Jacquet module of π_p). Assuming $\underline{\chi}$ is n-regular and numerically noncritical, that $r_p(\pi)(G_{F,v})^{ZC} \supset SL_2$ for all $v|p$, and that $\mathrm{Sym}^{n-1} r_p(\pi)$ is automorphic, we concluded that $\mathrm{Sym}^{n-1}\mathscr{C} \subset \mathscr{E}_n$, where $\mathscr{E}_n$ is the eigenvariety for G_n.

Now suppose $(\pi', \chi') \in RA_2 \cap \mathscr{C}$ is such that

- $r_p(\pi')(G_{F,v})^{ZC} \supset SL_2$ for all $v|p$,
- χ' is n-regular, and
- (π', χ') is generic.

The first condition implies that $\operatorname{Sym}^{n-1} r_p(\pi')|_{G_{F,v}}$ is irreducible for all $v|p$ (since $\operatorname{Sym}^{n-1} std$ is irreducible in characteristic 0). The second implies that

$$\operatorname{Sym}^{n-1}\chi' = ((\chi_1')^{n-1}, (\chi_1')^{n-2}\chi_2', \dots)$$

is regular. Finally, we can actually conclude that $(\operatorname{Sym}^{n-1} r_p(\pi'), \operatorname{Sym}^{n-1}\chi')$ is generic. (This is why we're switching to the generic condition from numerical noncriticality, which is not preserved by symmetric powers.) Consequently, $\operatorname{Sym}^{n-1}(\pi', \chi') \in RA_n$, and therefore $\operatorname{Sym}^{n-1} r_p(\pi')$ is automorphic. (That is to say, if you have one suitable classical point on a component of $\mathscr{E}_2$, which remains automorphic after taking the symmetric power, the same is true for "most" other classical points on the same component.)

Proof that $(\operatorname{Sym}^{n-1} r_p(\pi'), \operatorname{Sym}^{n-1}\chi')$ *is generic.* Let $v|p$ and assume that

$$WD(r_p(\pi')|_{G_{F_v}}) = \chi_1 \oplus \chi_2$$

with eigenbasis e_1, e_2, so that the Weil group acts by χ_i on e_i. (If $N \neq 0$ the argument is easier—go through it and see. $WD(r_p(\pi')|_{G_{F_v}})$ is always reducible because it is trianguline.) Let $\tau : F_v \hookrightarrow L$ and let the HT weights be $k_{\tau,1} < k_{\tau,2}$. We have

$$\operatorname{Fil}^i_\tau WD(r_p(\pi')|_{G_{F_v}}) = \begin{cases} \text{all} & i \leq k_{\tau,1} \\ \langle f_\tau \rangle & k_{\tau,1} > i \geq k_{\tau,2} \\ (0) & i > k_{\tau,2} \end{cases}$$

for some f_τ. Genericity is equivalent to saying that $f_\tau = \alpha_\tau e_1 + e_2$ for some $\alpha_\tau \neq 0$.

Then $\operatorname{Sym}^{n-1}(WD(r_p(\pi')|_{G_{F_v}}))$ has basis $e_1^i e_2^{n-1-i}$ for $i = 0, \dots, n-1$, and the irreducible WD subreps are indexed by $I \subset \{0, \dots, n-1\}$, corresponding to the submodule with basis $\{e_1^i e_2^{n-1-i} \mid i \in I\}$. Genericity is equivalent to saying that for all I,

$$\langle e_1^i e_2^{n-i} \mid i \in I\rangle \cap \langle (\alpha_\tau e_1 + e_2)^{n-1}, (\alpha_\tau e_1 + e_2)^{n-2} e_1, \dots, (\alpha_\tau e_1 + e_2)^{\#I} e_1^{n-1-\#I}\rangle = (0).$$

Shifting our linear combinations a bit, we can also write the second space being intersected above as

$$(\alpha_\tau e_1 + e_2)^{\#I} \otimes \langle e_2^{n-1-\#I}, \dots, e_1^{n-1-\#I}\rangle.$$

We can think of elements of the two vector spaces being intersected as polynomials, with $e_1 = X$ and $e_2 = 1$. Then it suffices to check that

$$\sum_{i\in I} a_i X^i \neq Q(X)(1+\alpha X)^{\#I}$$

for any a_i and polynomial Q, unless both sides are 0. We will show that if

$$\left(\frac{d}{dX}\right)^j \sum_{i\in I} a_i X^i \bigg|_{X=-\frac{1}{\alpha}} = 0$$

for $j = 0, \dots, \#I - 1$, then $a_i = 0$ for all i. The LHS is

$$\sum_{i\in I} (-\alpha)^j i(i-1)\cdots(i-j+1) a_i \left(-\frac{1}{\alpha}\right)^i.$$

This can be written as the following product of a matrix with a vector:

$$\begin{pmatrix} \cdots & 1 & \cdots \\ \cdots & i(-\alpha) & \cdots \\ \cdots & i(i-1)(-\alpha)^2 & \cdots \\ \cdots & i(i-1)(i-2)(-\alpha)^3 & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} a_{i_1}(-1/\alpha)^{i_1} \\ a_{i_2}(-1/\alpha)^{i_2} \\ \vdots \end{pmatrix}$$

where the columns of the matrix are indexed by I. We just need the determinant of this matrix to be nonzero. But by successive row operations we can make this matrix look like

$$\begin{pmatrix} \cdots & 1 & \cdots \\ \cdots & i(-\alpha) & \cdots \\ \cdots & (i(-\alpha))^2 & \cdots \\ \cdots & (i(-\alpha))^3 & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix}$$

which is a Vandermonde matrix, whose determinant is

$$\prod_{i_1, i_2 \in I, i_1 \neq i_2} (i_1(-\alpha) - i_2(-\alpha)) \neq 0.$$

□

19.2 The Coleman-Mazur eigencurve

So far we've been working with a general rank 2 unitary group over a general totally real field; now we want to work with the Coleman-Mazur eigencurve so that we can use the calculations of Buzzard and Kilford. The convention is that the weight space is $\mathscr{W}_0 = \mathscr{C}_{\mathbb{Z}_p^\times}$ and the character variety is $\mathscr{C}_0 = \mathscr{C}_{\mathbb{Q}_p^\times/\mathbb{Z}_p^\times \times \mathbb{Q}_p^\times}$ (so we're forcing one character to be unramified). Let $\mathbb{T}_0(M)$ be the $\mathcal{O}$-subalgebra of $\mathrm{End}_{\mathcal{O}}(H^1(X_1(M), \mathcal{O}))$ generated by T_q, S_q for $q \nmid Mp$. For $(N, p) = 1$, let $\mathbb{T}(N) = \varprojlim_r \mathbb{T}_0(Np^r)$, a complete semilocal noetherian ring.

The theory that we previously described for 0-dimensional Shimura varieties coming from unitary groups compact at infinity still works well in this setting, because H^0 and H^2 are trivial to understand and so we can pull out H^1. Let

$$X_0^{Hecke} = (\mathrm{Spf}\, \mathbb{T}(N))^{an}.$$

The Coleman-Mazur eigencurve $\mathscr{E}_0 \subset X_0^{Hecke} \times \mathscr{C}_0$ is a curve living over $\mathscr{W}_0$, a finite union of open unit discs. As usual, $\mathscr{E}_0 \to \mathscr{W}_0$ is locally finite on $\mathscr{E}_0$ and every irreducible component has Zariski open image (misses a discrete set of points). $\mathscr{E}_0$ is Zariski closed in $X_0^{Hecke} \times \mathscr{C}_0$ and if $(x, \underline{\delta}) \in \mathscr{E}_0$ then $r_x|_{G_{\mathbb{Q}_p}}$ is trianguline. If $\underline{\delta}$ is numerically noncritical or $r_x|_{G_{\mathbb{Q}_p}}$ is generic, it is trianguline with parameter $\underline{\delta}$.

Let RA_0 be the set of (π, χ) such that

- π is a PRAC automorphic representation of $GL_2(\mathbb{Q})$ (basically, a modular form of weight $k \geq 2$),

- $\pi^{p,\infty,U_1(N)} \neq (0)$,
- and $\underline{\chi} = (\chi_1, \chi_2)$ is a smooth character of $\mathbb{Q}_p^\times$ with χ_1 unramified, such that
- $\mathbb{C}(\chi_1 |\cdot|_p, \chi_2) \hookrightarrow J_B(\pi_p)$, and
- π_∞ has the same infinitesimal character as $(\mathrm{Sym}^{k-2}\,\mathbb{C}^2)^\vee$ for some $k \in \mathbb{Z}_{\geq 2}$.

In this setup the central character is fixed, so we have $v_p((\chi_1\chi_2)(p)) = k-1$ and $HT(r_p(\pi)|_{G_{\mathbb{Q}_p}}) = \{0, k-1\}$. The Newton-above-Hodge condition implies $v_p(\chi_i(p)) \geq 0$ for $i = 1, 2$. $(\pi, \underline{\chi})$ being numerically noncritical is equivalent to $v_p(\chi_1(p)) \neq k-1$, or $v_p(\chi_2(p)) \neq 0$.

We have a map

$$\begin{aligned} RA_0 &\to \mathscr{E}_0 \\ (\pi, \chi) &\mapsto (\theta_\pi, (\chi_1, \chi_2\tau^{1-k})) \end{aligned}$$

where θ_π is the character of the Hecke algebra on $(\pi^{p,\infty})^{U_1(N)}$ and $\tau : \mathbb{Q}_p \hookrightarrow L$ is the standard embedding. ($\mathscr{E}_0$ is in fact defined as the closure of these points.) Now we can state the same kind of theorem as before for this eigencurve.

Theorem 19.2.1. *Suppose $(\pi, \underline{\chi}) \in RA_0$ (the classical terminology is that this is a "p-stabilized" form, meaning that it's an eigenform for U_p in addition to the Hecke operators away from p) is n-regular and numerically noncritical, that $r_p(\pi)(G_{\mathbb{Q}_p})^{ZC} \supset SL_2$, and that* $\mathrm{Sym}^{n-1}(r_p(\pi))$ *is automorphic. Let $\mathscr{C}$ be an irreducible component of $\mathscr{E}_2$ containing $(\pi, \underline{\chi})$ and let $(\pi', \underline{\chi}') \in RA_0 \cap \mathscr{C}$ be such that $\underline{\chi}'$ is n-regular and $r_p(\pi')(G_{\mathbb{Q}_p})^{ZC} \supset SL_2$. Then* $\mathrm{Sym}^{n-1}(r_p(\pi'))$ *is also automorphic.*

Proof. Let $\mathscr{E}_0 \hookrightarrow X_1^{Hecke} \times \mathscr{C}_0$. We are going to move to the unitary group, for which we may have to twist the Galois representation by a character. It turns out that we can find a CM abelian extension $F/\mathbb{Q}$, a finite etale cover $\tilde{\mathscr{E}}_0 \to \mathscr{E}_0$ with some conditions we'll skip over, a character $\psi : G_F \to \mathscr{O}_{\tilde{\mathscr{E}}_0}^\times$, and a map

$$\begin{aligned} \tilde{\mathscr{E}}_0 &\to X_{2,F}^{univ} \times \mathscr{C}_{2,F}^{csd} \\ \tilde{x} &\mapsto (r_x|_{G_F} \otimes \psi, \underline{\delta}\psi) \end{aligned}$$

where $\tilde{x} \in \tilde{\mathscr{E}}_0$ goes to $(x, \underline{\delta}) \in \mathscr{E}_0 \hookrightarrow X_1^{Hecke} \times \mathscr{C}_0$. (Note that ψ can be chosen to force conjugate self-duality in the target.) We have $\mathscr{E}_{2,F} \subset X_{2,F}^{univ} \times \mathscr{C}_{2,F}^{csd}$, and using base change for $F/\mathbb{Q}$, we find that actually $\tilde{\mathscr{E}}_0$ goes into $\mathscr{E}_{2,F}$ by checking on the Zariski dense set of classical points. Then we apply the result for unitary groups, and we just need to check genericity, which we are no longer assuming.

For this, consider the case $WD(r_p(\pi')|_{G_{\mathbb{Q}_p}}) = \chi_1 \oplus \chi_2$ (again the non-semisimple case is easier—exercise). This is generic unless Fil_τ^{k-1} is χ_1 or χ_2; WLOG suppose it is χ_1. By weak admissibility, we have $v_p(\chi_1(p)) \geq k-1$, $v_p(\chi_2(p)) \geq 0$, and $v_p((\chi_1\chi_2)(p)) = k-1$, so these are all equalities, so χ_1 and χ_2 are both admissible subspaces, so $r_p(\pi) = \chi_2 \oplus (\chi_1|\cdot|^{k-1})\epsilon_p^{1-k}$ is reducible. But this is not true. (Basically, in 2 dimensions, assuming irreducibility, genericity is forced.) ∎

19.3 The case $p = 2, N = 1$

Buzzard-Kilford made an explicit calculation of $\mathscr{E}_0$ in this situation. In this case, we have the following decomposition $\mathscr{W} = \mathscr{W}^+ \coprod \mathscr{W}^-$: the points of $\mathscr{W}$ are continuous characters of $\mathbb{Z}_2^\times = \{\pm 1\} \times (1 + 4\mathbb{Z}_2)$ (in which $1 + 4\mathbb{Z}_2 = \langle 5 \rangle \cong \mathbb{Z}_2$), and $\delta_2(-1)$ is 1 on $\mathscr{W}^+$ and -1 on $\mathscr{W}^-$. Because the determinant of the corresponding Galois representation is required to be -1, and we can't get a sign from a tame prime because there are none, $\mathscr{E}_0$ lives over only $\mathscr{W}^-$, which is an open unit disc via the map

$$\begin{aligned} \mathscr{W}^- &\to \Delta[0, 1) \\ \underline{\delta} &\mapsto (5\delta_2(5))^{-1} - 1 =: w(\underline{\delta}). \end{aligned}$$

(Warning: this is a different normalization from the one in Buzzard-Kilford.)

Theorem 19.3.1 (Buzzard-Kilford)**.** *Let $\mathscr{W}^-(b) \subset \mathscr{W}^-$ be the annulus $\Delta(1/8, 1)$ and $\mathscr{E}_0(b)$ the part of $\mathscr{E}_0$ over $\mathscr{W}^-(b)$. Then*

$$\mathscr{E}_0(b) = \coprod_{i=1}^{\infty} \mathscr{E}_0(b)_i$$

where $\mathscr{E}_0(b)_i \xrightarrow{\sim} \mathscr{W}^-(b)$ for each i. Furthermore, the map

$$\begin{aligned} s : \mathscr{E}_0(b)_i \to \mathscr{C}_0 \to \mathbb{G}_m^{an} &\xrightarrow{v} \mathbb{Q} \\ x \mapsto (\delta_1, \delta_2) \mapsto \delta_1(2) &\mapsto v(\delta_1(2)), \end{aligned}$$

together with this isomorphism, induces

$$\begin{aligned} \mathscr{W}^-(b) &\to \mathbb{Q} \\ w &\mapsto iv_2(w). \end{aligned}$$

Note that this implies that the slopes go to 0 as you go toward the boundary. Note also that the indexing starts at 1, not 0, because we are leaving out the ordinary Eisenstein component (because earlier we constructed the eigencurve using H^1 of the completed modular curve). There are some similar results for higher primes and tame levels but they are not as precise.

19.4 Twins

Let $(\pi, (\chi_1, \chi_2)) \in RA_0$ be such that π_∞ has the same infinitesimal character as $(\mathrm{Sym}^{k-1}\, \mathbb{C}^2)^\vee$. Assume π_p is a principal series (what we are about to say doesn't work for twists of Steinberg representations). Then we have a second point

$$(\pi \otimes \chi_\pi^{-1} \| \cdot \|^{2-k}, (\chi_1^{-1} | \cdot |_p^{1-k}, \chi_2^{-1} | \cdot |_p^{1-k})) \in RA_0$$

(because the Jacquet module contains two characters, one of which is the swapped version of the other up to normalizations, but you have to twist to keep the second component unramified). Let τ be the "twin" map associating one of these points to the other; it is an involution. We have

$$s(\tau(\pi,\underline{\chi})) + s(\pi,\underline{\chi}) = k-1$$
$$w(\pi,\chi) = 5^{k-2}\chi_2(5)^{-1} - 1$$
$$w(\tau(\pi,\chi)) = 5^{k-2}\chi_2(5) - 1.$$

Note that $w(\pi,\chi)$ and $w(\tau(\pi,\chi))$ have the same 2-adic valuation, because $\chi_2(5)$ and $\chi_2(5)^{-1}$ are the same distance from 1. So if $(\pi,\underline{\chi}) \in \mathscr{E}_0(b)_i$ and $\tau(\pi,\underline{\chi}) \in \mathscr{E}_0(b)_{i'}$, then

$$(k-1) = (i+i')v_2(w(\pi,\underline{\chi})).$$

Here are a few more observations about $p=2, N=1$. In this case $(\pi,\underline{\chi}) \in RA_0$ is automatically numerically noncritical and $r_p(\pi)|_{G_{\mathbb{Q}_p}}$ irreducible, because violating these requires having an ordinary component of slope 0, and $\mathscr{E}_0$ for $p=2, N=1$ does not (we left out the Eisenstein component). Furthermore, $(r_2(\pi)|_{G_{\mathbb{Q}_2}})^{ZC} \supset SL_2$, because otherwise we would have $r_2(\pi)|_{G_{\mathbb{Q}_2}} = \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}_2}} \psi$ for some $K/\mathbb{Q}_2$ and character ψ, so that $WD(r_2(\pi)|_{G_{\mathbb{Q}_2}}) = \mathrm{Ind}_{W_K}^{W_{\mathbb{Q}_2}} \psi|_{W_K}$, but this is reducible because it is trianguline. So the induction decomposes, which means we must have $\psi|_{W_K}^{\tau} = \psi|_{W_K}$ where $\mathrm{Gal}(K/\mathbb{Q}_2) = \{1,\tau\}$, so $\psi^\tau = \psi$ since the Weil group is dense in the Galois group, so $r_2(\pi)|_{G_{\mathbb{Q}_2}}$ is reducible, but as we just checked, it is not.

Corollary 19.4.1. *If $(\pi,\chi), (\pi',\chi')$ are in the same component of $\mathscr{E}_0$ for $p=2$, $N=1$, χ,χ' are n-regular, and $\mathrm{Sym}^{n-1} r_p(\pi)$ is automorphic, then $\mathrm{Sym}^{n-1} r_p(\pi')$ is automorphic.*

19.5 Conclusion

Lemma 19.5.1. *Suppose π is an everywhere unramified PRAC automorphic representation of $GL_2(\mathbb{A})$ and π_∞ has the same infinitesimal character as $(\mathrm{Sym}^{k-2}\mathbb{C}^2)^\vee$. Suppose that $\underline{\chi}$ is a refinement of π_2. Then $\underline{\chi}$ is n-regular for all n.*

Proof. Since the WD rep is everywhere unramified, we can write $WD(r_2(\pi)|_{G_{\mathbb{Q}_2}}) = (\mathrm{Frob}_2)$, where Frob_2 has eigenvalues α,β. We need to know that α/β is not a root of unity.

Fact (see e.g. Serre [10]): if f is a newform of weight k and level Np^r with N prime to p, then there is a newform f' of level dividing N and weight k' with $2 \le k' \le p+1$ such that $\overline{r_p(f)} \cong \overline{r_p(f')} \otimes \chi$ for some character χ. So the ratio of the eigenvalues of $\overline{r_p(f)}(\sigma)$ is congruent to the ratio of the eigenvalues of $\overline{r_p(f')}(\sigma)$ mod a prime above p. So if we want to know what the possible ratios are for $r_p(f)$, we only have to look at forms of weight between 2 and $p+1$ and level dividing N.

For $N=1$ and $p<11$, the only such forms are Eisenstein series, and we know what their associated residual representations look like: depending on normalization, we have $r_p(E_k) = 1 \oplus \epsilon_p^{1-k}$ where E_k is the level 1 Eisenstein series of weight k. So

$$\overline{r}_p(E_k)(\mathrm{Frob}_2) \equiv \mathrm{diag}(1, 2^{k-1}) \pmod{\text{prime above } p}.$$

For example, for $p=3$, the possibilities are E_2 and E_4, for both of which $r_3(f)(\mathrm{Frob}_2) \equiv \mathrm{diag}(1,-1)$. Therefore, up to twist, we must have $\overline{r_3(\pi)}^{ss} = 1 \oplus \overline{\epsilon}_3$, and $\alpha/\beta \equiv -1$ at a prime above 3. Since reduction mod 3 is an injection on roots of unity of order prime to 3, if α/β is a root of unity, it has to be -1 times a 3-power order root of 1. But similarly, using 5 instead of 3, we find that α/β must be $\pm i$ times a 5-power order root of unity. Nothing is both, so this is impossible. $\square$

So now we can drop the n-regularity assumption too.

Now suppose (π_1, χ_1) and (π_2, χ_2) are in RA_0 with π_1, π_2 unramified everywhere. Then (π_j, χ_j) will be on the same irreducible component of $\mathscr{E}_0$ as some $\mathscr{E}_0(b)_{i(j)}$ (since each irreducible component has Zariski open image). Choose $z_j \in \mathscr{E}_0(b)_{i(j)}$ with

$$w(z_j) = 5^{2i(j)+2^{m+1}-3} - 1$$

(which has $v_2 = 2$) for sufficiently large m to be determined later. Let $z_j = (\pi'_j, \chi'_j)$ and let $\tau z_j = (\pi''_j, \chi''_j)$. We have $\tau z_j \in \mathscr{E}(b)_{i(j)'}$ where

$$i(j)' = \frac{2i(j) + 2^{m+1} - 2}{2} - i(j) = 2^m - 1.$$

So on the $i(1)$-component we have (π_1, χ_1) and (π'_1, χ'_1), the latter over weight $5^{2i(1)+2^{m+1}-3} - 1$. On the $(2^m - 1)$-component we have $\tau(\pi'_1, \chi'_1) = (\pi''_1, \chi''_1)$. On the $i(2)$-component we have (π_2, χ_2) and (π'_2, χ'_2), the latter over weight $5^{2i(2)+2^{m+1}-3} - 1$. Then $\tau(\pi'_2, \chi'_2) = (\pi''_2, \chi''_2)$ is also on $2^m - 1$.

If the symmetric power of a point is automorphic then the symmetric power of its twin is also automorphic, since one is just a twist of the other. So $\mathrm{Sym}^{n-1}(\pi_1, \chi_1)$ is automorphic if and only if $\mathrm{Sym}^{n-1}(\pi'_1, \chi'_1)$ is, iff $\mathrm{Sym}^{n-1}(\pi''_1, \chi''_1)$ is, iff $\mathrm{Sym}^{n-1}(\pi''_2, \chi''_2)$ is, iff $\mathrm{Sym}^{n-2}(\pi'_2, \chi'_2)$ is, iff $\mathrm{Sym}^{n-1}(\pi_2, \chi_2)$ is.

We conclude our main theorem.

Theorem 19.5.2. *If one level one elliptic modular newform of weight $k \geq 2$ has an $(n-1)$st symmetric power lift to $GL_n(\mathbb{A})$, then all such forms do.*

(Technically we should make sure that if we have two points on $\mathscr{E}_0$ on the same component, and we choose lifts on $\tilde{\mathscr{E}}_0$, we can choose them so that they end up on the same component in $\mathscr{E}_{2,F}$. We can in fact do that.)

The rest of the paper is constructing one such form.

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