Math 249C, Spring 2021: potential automorphy over CM fields

Course by Richard Taylor

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Abstract

Richard Taylor taught a course on potential automorphy over CM fields at Stanford in Spring 2021.

These are scribed notes from the course. Conventions are as follows: Each lecture gets its own "chapter," and appears in the table of contents with the date.

Of course, these notes are not a faithful representation of the course, either in the mathematics itself or in the quotes, jokes, and philosophical musings; in particular, the errors are our fault. By the same token, any virtues in the notes are to be credited to the lecturer and not the scribe(s).[1] Please email suggestions to lynnelle@stanford.edu.

Contents

[1] This abstract has been adapted from Akhil Mathew's introduction to his notes, with his permission.

1 March 30: introduction.

We want to generalize arithmetic results we know about modular forms to settings where there's less obviously any number theory involved.

1.1 Context

Here are two well-known theorems.

Theorem 1.1.1 (Ramanujan conjecture, theorem of Deligne)**.** *If a is an eigenvalue of T_p on $H^1(X_1(N), \mathbb{C})$, for $p \nmid N$, then $|a| \leq 2\sqrt{p}$.*

Theorem 1.1.2 (Shimura-Taniyama conjecture)**.** *If $E/\mathbb{Q}$ is an elliptic curve, then $L(E, s)$ (initially defined for $\Re(s) \geq 2$) has holomorphic continuation to $\mathbb{C}$ and satisfies*

$$(2\pi)^{s-2}\Gamma(s-2)L(E, 2-s) = \pm(2\pi)^{-s}\Gamma(s)L(E, s).$$

The Ramanujan Conjecture proof strategy is roughly as follows. Given $\mathbb{C} \cong \overline{\mathbb{Q}}_l$, we have

$$H^1(X_1(N), \mathbb{C}) \cong H^1_{et}(X_1(N)/\overline{\mathbb{Q}}, \overline{\mathbb{Q}}_l)$$

and the RHS has an action by $\mathrm{Gal}(\mathbb{Q}_{\{Nl\}}/\mathbb{Q})$. We have

$$T_p = \mathrm{Frob}_p + pS_p \, \mathrm{Frob}_p^{-1}$$

where $S_p^{\varphi(N)} = 1$ and S_p commutes with T_p, Frob_p. The eigenvalues α of Frob_p satisfy the Weil conjecture/theorem $|\alpha| = p^{1/2}$ (from algebraic geometry), so the eigenvalues of T_p are of the form $a = \alpha + p\zeta\alpha^{-1}$ where ζ is a root of unity (from S_p), and we conclude that

$|a| \leq 2p^{1/2}$. We are using that $X_1(N)$ is an algebraic variety defined over a number field (in fact $\mathbb{Q}$).

The Shimura-Taniyama Conjecture proof strategy is roughly as follows. We prove that there is a weight 2 modular form f for $\Gamma_0(N)$, where N is the conductor of E, such that $L(E, s) = L(f, s)$. i.e. $\mathrm{tr}(\mathrm{Frob}_p | T_l E)$ is the eigenvalue of T_p on f for all $p \nmid Nl$. This breaks down into the following two steps.

1. Find f_0 such that $\mathrm{tr}(\mathrm{Frob}_p | E[l])$ is congruent to the eigenvalue of T_p on f_0 for all $p \nmid Nl$ (for $l = 3, 5$).

2. Deduce the same thing modulo higher and higher powers of l and produce f in the limit.

Again, this crucially uses algebraic geometry and Galois theory.

1.2 Statements of theorems for this class

Suppose $F/\mathbb{Q}$ is an imaginary quadratic field. We are going to prove the analogue of the Shimura-Taniyama conjecture.

Theorem 1.2.1. *Suppose E/F is an elliptic curve. Then $L(E, s)$ has meromorphic continuation to $\mathbb{C}$ and satisfies*

$$(2\pi)^{2(s-2)}\Gamma(2-s)^2 L(E, 2-s) = \pm(2\pi)^{-2s}\Gamma(s)^2 L(E, s).$$

We are also going to prove the analogue of the Ramanujan conjecture, for which we need the analogue of the modular curve. Let

$$\mathbb{H}_3 = \mathbb{C} \times \mathbb{R}_{>0} = \{z + yj \mid z \in \mathbb{C} = \mathbb{R} \oplus i\mathbb{R}, y \in \mathbb{R}_{>0}\}$$

embedded inside the Hamiltonian quaternions. This is hyperbolic 3-space. It is acted on by $SL_2(\mathbb{C})$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\tau) = (a\tau + b)(c\tau + d)^{-1}.$$

(Since quaternions are not commutative, the order matters.) Since -1 acts trivially, this action factors through $PSL_2(\mathbb{C}) = PGL_2(\mathbb{C})$, hence we also get an action of $GL_2(\mathbb{C}) \twoheadrightarrow PGL_2(\mathbb{C})$ (but the induced action of $GL_2(\mathbb{C})$ doesn't satisfy the above formula; you have to normalize it into $SL_2(\mathbb{C})$ to calculate it).

We're going to set $F = \mathbb{Q}(i)$ for the moment. If $\mathfrak{n} \subset \mathbb{Z}[i]$ is an ideal, let

$$\Gamma_1(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}[i]) \mid c, d-1 \in \mathfrak{n} \right\}.$$

Then $\Gamma_1(\mathfrak{n})\backslash\mathbb{H}_3$ is a hyperbolic 3-manifold (also called a Bianchi manifold). For $\mathfrak{p} \subset \mathbb{Z}[i]$ a prime ideal, if $\mathfrak{p} = (\pi)$ (since $\mathbb{Z}[i]$ is a PID), we have a double coset

$$T_{\mathfrak{p}} = \left[\Gamma_1(\mathfrak{n}) \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} \Gamma_1(\mathfrak{n})\right]$$

which does not depend on the choice of π. Take the "interior cohomology"

$$H^i_{int}(\Gamma_1(\mathfrak{n})\backslash\mathbb{H}_3, \mathbb{C}) = \mathrm{im}(H^i_c(\Gamma_1(\mathfrak{n})\backslash\mathbb{H}_3, \mathbb{C}) \to H^i(\Gamma_1(\mathfrak{n})\backslash\mathbb{H}_3, \mathbb{C}))$$

for $i = 1, 2$ (the only cohomologies that are interesting, which are dual under Poincare duality). This is an ad hoc construction that we need as a substitute for compactifying the modular curve. It has an action of $T_{\mathfrak{p}}$.

Theorem 1.2.2. *If a is an eigenvalue of $T_{\mathfrak{p}}$ ($\mathfrak{p} \nmid \mathfrak{n}$) on $H^1_{int}(\Gamma_1(\mathfrak{n})\backslash\mathbb{H}_3, \mathbb{C})$ then*

$$|a| \leq 2(\#\mathbb{Z}[i]/\mathfrak{p})^{1/2}.$$

Here are some differences between these theorems and the previous ones.

1. $X_1(N)$ is the complex points of an algebraic variety defined over $\mathbb{Q}$, but $\Gamma_1(N)\backslash\mathbb{H}_3$ is not the complex points of any algebraic variety (for one thing it has odd real dimension). This takes away many of the strategies we used before, like the action of Frobenius on the étale cohomology of algebraic varieties, and the Galois representations associated to cohomology classes.

2. $H^i(X_1(N), \mathbb{C})$ or $H^i(X_1(N), \mathbb{Z}/l^r)$ is negligible except for $i = 1$ (as a compact manifold, $X_1(N)$ has a 1-dimensional H^0 and H^2, but they're easy). This means that you can predict the dimension of H^1 and the eigenvalues of T_p using the Euler characteristic and Selberg/Lefschetz trace formulas ("the Euler characteristic sees everything").

   Also, $H^1(X_1(N), \mathbb{Z}_l)$ is torsion-free. This is because we have an exact sequence

   $$0 \to H^i(\mathbb{Z}_l)/l^n \to H^i(\mathbb{Z}_l/l^n) \to H^{i+1}(\mathbb{Z}_l)[l^n] \to 0$$

   and putting $i = 0$, since H^0 is negligible, we see that the last term, which gives the torsion in H^1, goes away. Furthermore, putting $i = 1$, we see that the cohomology with torsion coefficients can be obtained from the cohomology with integer coefficients by reducing mod l.

   On the other hand, $H^i(\Gamma_1(\mathfrak{n}), \mathbb{H}_3, \mathbb{C})$ (or $\mathbb{Z}/l^r$) is negligible except for $i = 1, 2$ and is "equally complicated" in those degrees by Poincare duality. This means that the Euler characteristic sees nothing, and the dimension behaves erratically as $\mathfrak{n}$ varies (experiments say it is frequently 0 for split primes).

   Furthermore, experiments show that $H^2(\Gamma_1(\mathfrak{n})\backslash\mathbb{H}_3, \mathbb{Z})$ has lots of torsion ("enormous primes with 20-30 digits" even for small $\mathfrak{n}$, for no obvious good reason, though not very many of them at a time). And knowing the cohomology for $\mathbb{Z}$ doesn't tell you what it is for $\mathbb{Z}/l^n$.

3. Let $\overline{r} : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_l)$. To simplify, assume $\overline{r}|_{G_{\mathbb{Q}_l}}$ is absolutely irreducible and Fontaine-Laffaile with HT numbers $\{0, 1\}$, $\det \overline{r} = \epsilon_l^{-1}$ (where ϵ_l is the cyclotomic character), $l > 3$, and $\overline{r}(c) \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. (Similar phenomena hold without these assumptions but would take longer to state.) We have commuting surjections

   $$R^{univ}_{\overline{r}|_{G_{\mathbb{Q}_l}}} \twoheadrightarrow R^{univ,FL}_{\overline{r}|_{G_{\mathbb{Q}_l}}} \twoheadrightarrow R^{univ,FL}_{\overline{r}}$$

$$R^{univ}_{\overline{r}|_{G_{\mathbb{Q}_l}}} \twoheadrightarrow R^{univ}_{\overline{r}} \twoheadrightarrow R^{univ,FL}_{\overline{r}}.$$

(According to the Fontaine-Mazur conjecture, points on the space coming from $R^{univ,FL}_{\overline{r}}$ should correspond to modular forms of weight 2.) Taking the generic fiber of these gives the rigid spaces

$$X^{univ}_{\overline{r}|_{G_{\mathbb{Q}_l}}} \hookleftarrow X^{univ,FL}_{\overline{r}|_{G_{\mathbb{Q}_l}}} \hookleftarrow X^{univ,FL}_{\overline{r}}$$

$$X^{univ}_{\overline{r}|_{G_{\mathbb{Q}_l}}} \hookleftarrow X^{univ}_{\overline{r}} \hookleftarrow X^{univ,FL}_{\overline{r}}.$$

(These are all with fixed determinant ϵ_l^{-1}.) It turns out that $X^{univ}_{\overline{r}|_{G_{\mathbb{Q}_l}}}$ has dimension 3 (by computing the dimension of the tangent space as an H^1 using Euler characteristics and the dimension of the cohomology of the trace zero part of the adjoint representation). $X^{univ,FL}_{\overline{r}|_{G_{\mathbb{Q}_l}}}$ has dimension 1 by Fontaine-Laffaile theory. $X^{univ}_{\overline{r}}$ has dimension $3 - \dim(\mathrm{ad}^0\,\overline{r})^{G_{\mathbb{R}}} = 2$ (because $\mathrm{ad}\,\overline{r}$ has two 1 eigenvalues and two -1 eigenvalues, and one of the 1s comes from the diagonal, so $\mathrm{ad}^0\,\overline{r}$ has one 1 and two -1s, so the $G_{\mathbb{R}}$-fixed part is 1-dimensional). Since $X^{univ,FL}_{\overline{r}}$ is the intersection of $X^{univ,FL}_{\overline{r}|_{G_{\mathbb{Q}_l}}}$ and $X^{univ}_{\overline{r}}$, one might guess that it is 0-dimensional, so that there are finitely many FL characteristic zero l-adic liftings of $\overline{r}$, that they correspond to the finitely many classical Hecke eigenforms of a given level in weight 2, that $R^{univ,FL}_{\overline{r}}$ corresponds to the classical Hecke algebra, and that the intersections are transverse.

On the other hand suppose we look at $\overline{r} : G_{\mathbb{Q}(i)} \to GL_2(\mathbb{F}_l)$. Assume that l is inert in $\mathbb{Q}(i)$, $\overline{r}|_{G_{\mathbb{Q}(i)_l}}$ is absolutely irreducible and FL with HT numbers $\{0, 1\}$, $\det \overline{r} = \epsilon_l^{-1}$, and $l > 3$. We get

$$X^{univ}_{\overline{r}|_{G_{\mathbb{Q}(i)_l}}} \supset X^{univ,FL}_{\overline{r}|_{G_{\mathbb{Q}(i)_l}}} \leftarrow X^{univ,FL}_{\overline{r}}$$

$$X^{univ}_{\overline{r}|_{G_{\mathbb{Q}(i)_l}}} \leftarrow X^{univ}_{\overline{r}} \supset X^{univ,FL}_{\overline{r}}$$

where $X^{univ}_{\overline{r}|_{G_{\mathbb{Q}(i)_l}}}$ is 6-dimensional (because we multiply by the degree of the field extension), $X^{univ,FL}_{\overline{r}|_{G_{\mathbb{Q}(i)_l}}}$ is 2-dimensional by FL theory, $X^{univ}_{\overline{r}}$ is 3-dimensional (from $6 - \dim \mathrm{ad}^0(\overline{r})^{G_{\mathbb{C}}}$, because $G_{\mathbb{C}}$ is trivial). So we might expect that $X^{univ,FL}_{\overline{r}}$ is empty, but that is not always true: if we take a suitable elliptic curve over $\mathbb{Z}[i]$ its Tate module (or dual) will provide a point.

In the integral version of this, the dimensions are all larger by 1: the universal local deformation is 7-dimensional, inside which we intersect the 3-dimensional Fontaine-Laffaile local deformations and the 4-dimensional global deformations, and so one would expect $R^{univ,FL}_{\overline{r}}$ to be 0-dimensional (with intersections only happening at the special point). This is very often correct, because torsion in the cohomology gives places where the maximal ideal of the Hecke algebra is also a minimal ideal, so that the Hecke algebra is 0-dimensional. But the "important" case is when it is 1-dimensional and we have a non-transverse intersection.

For example, suppose $H^2(\Gamma_0(\mathfrak{n})\backslash\mathbb{H}_3, \mathbb{Z}_l) \cong \mathbb{Z}/l^2$. Then $\mathbb{T}_0(\mathfrak{n}) \subset \mathrm{End}(H^2(\Gamma_0(\mathfrak{n})\backslash\mathbb{H}_3, \mathbb{Z}_l))$ is a finite-length $\mathbb{Z}_l$-module, i.e. the Krull dimension of $\mathbb{T}_0(\mathfrak{n})$ is 0. On the other hand if $H^1(X_0(N), \mathbb{Z}_l) \cong \mathbb{Z}_l$ then the Krull dimension of $\mathbb{T}_0(\mathfrak{n})$ is 1, which would match with our expectation that if you invert l you get something 0-dimensional but on the integral side you get one more dimension, like with the classical Hecke algebra and modular curves.

The solution to all of these problems is to work consistently with entire complexes rather than just H^1.

1.3 Locally symmetric spaces

In the last few topics courses here, we've always put ourselves in the situation of a reductive algebraic group $H/\mathbb{Q}$ where $H(\mathbb{R})$ is compact mod center, so that $H(\mathbb{Q})\backslash H(\mathbb{A})/H(\mathbb{R})U$ is a finite set, and automorphic forms on H are functions on a finite set. Then for example it's easy to work integrally, because we just take integer-valued functions on such a finite set.

Heuristically, it is possible to use functoriality to switch to such an H (even if you started with e.g. GL_n) when you are interested in automorphic forms which are regular algebraic (the infinitesimal characters are integers, not random complex numbers) and (vaguely speaking) conjugate self-dual ($\pi^c \sim \pi^\vee$, maybe after a twist). This is the same as what you need on the Galois side for the intersection dimensions to behave. We will not have this for $GL_2(\mathbb{Q}(i))$; then representations associated to modular forms are essentially self-dual, so to be conjugate self-dual they have to be isomorphic to their conjugates, thus the elliptic curve descends to $\mathbb{Q}$. So if your elliptic curve isn't isogenous to its complex conjugate this condition will not be satisfied.

So we will really need to work with automorphic forms not on a finite set, where $G(\mathbb{Q})\backslash G(\mathbb{A})/UU_\infty$ is a locally symmetric space with U_∞ the maximal compact mod center subgroup of $G(\mathbb{R})$. How? Sometimes this locally symmetric space is an algebraic variety (then if you define it integrally you get integral automorphic forms), but in this case we generally could have switched to H with $H(\mathbb{R})$ compact mod center anyway. If not, we just have to work integrally with a locally symmetric space and use its singular cohomology.

So our first focus will be locally symmetric spaces and their cohomology. For this we will follow Borel-Serre, "corners and arithmetic groups" [3], though it is not a perfect reference.

Let $G/\mathbb{Q}$ be a connected linear algebraic group. (Borel-Serre don't assume connectedness but we'll stick to that case. Note we are not assuming that G is reductive! We will need to look at the boundary, coming from parabolic subgroups, which are not reductive.) Let N_G be the unipotent radical of G and $L_G = G/N_G$ (the reductive part of G). $G \twoheadrightarrow L_G$ has a splitting $\tilde{L} \subset G$, a maximal reductive subgroup, which is not unique, but all splittings $\tilde{L}/k$ are conjugate by $N_G(k)$ (for k characteristic 0). Let $Z(L_G)$ be the center of L_G. Let $A_G \subset Z(L_G)$ be the maximal split torus. Let $R_S(G)$ be the inverse image of A_G in G, so we have an exact sequence

$$0 \to N_G \to R_S(G) \to A_G \to 0.$$

Let

$$M_G = \bigcap_{\chi \in X^*(G)(\mathbb{Q})} \ker(\chi^2)$$

(i.e. χ runs over homomorphisms $\chi : G \to \mathbb{G}_m$ defined over $\mathbb{Q}$). M_G may not be connected. If $\tilde{L} \subset G$ lifts L_G, with $\tilde{A} \subset \tilde{L}$ corresponding to A_G, we have $\tilde{A} \ltimes M_G \twoheadrightarrow G$. On the real points this becomes

$$\tilde{A}(\mathbb{R})^+ \ltimes M_G(\mathbb{R}) \xrightarrow{\sim} G(\mathbb{R}).$$

(Here if $H/\mathbb{R}$ then by $H(\mathbb{R})^+$ we mean the connected component of the identity in $H(\mathbb{R})$ WRT the real topology.)

Example 1.3.1. Let

$$G = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \in GL_3 \right\}.$$

Then

$$N_G = \left\{ \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \in GL_3 \right\}.$$

We have $L_G = GL_2 \times GL_1$, $A_G \cong \mathbb{G}_m^2$, and $M_G = (SL_2^\pm \times \{\pm 1\}) \ltimes N_G$, where $SL_2^\pm = \{g \in GL_2 \mid \det g = \pm 1\}$. Any $g \in G(\mathbb{R})$ can be written as a product of something of the form

$$\mathrm{diag}(\lambda, \lambda, \mu) \in A_G(\mathbb{R})^+$$

where $\lambda, \mu \in \mathbb{R}_{>0}$, and something of the form

$$\begin{pmatrix} A & * \\ 0 & b \end{pmatrix} \in M_G(\mathbb{R})$$

where $b \in \{\pm 1\}$ and A is a 2×2 matrix such that $\det A = \pm 1$.

Next, parabolic subgroups. We call an algebraic subgroup $P \subset G$ parabolic if G/P is projective. (In particular, P contains N_G.) There is a bijection between parabolics in G and parabolics in L_G. If P is parabolic and $Q \supset P$, then Q is parabolic. If $P_0 \subset G$ is a minimal parabolic defined over k of characteristic 0, then any parabolic over k is $G(k)$-conjugate to a unique parabolic containing P_0. If $P \subset G$ is parabolic, then $A_G \subset A_Q$, A_Q/A_G is a split torus, and $\dim A_Q/A_G$ equals the number of maximal proper parabolics containing Q. If these are $P_1, \ldots, P_r$, then we have

$$\prod_i A_{P_i}/A_G \twoheadrightarrow A_Q/A_G$$

with finite kernel. "The chain of parabolics is discrete, and every time you go down one in the chain, the dimension of A_Q goes up by 1."

Example 1.3.2. If $G = GL_3$, then we can choose

$$P_0 = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in GL_3 \right\}.$$

It is contained in

$$P_1 = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \in GL_3 \right\}, \qquad P_2 = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in GL_3 \right\}.$$

Then

$$N_{P_0} = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \in GL_3 \right\},$$

$$N_{P_1} = \left\{ \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \in GL_3 \right\}, \qquad N_{P_2} = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in GL_3 \right\},$$

and furthermore

$$L_{P_0} = \mathbb{G}_m^3, \qquad L_{P_1} = GL_2 \times GL_1, \qquad L_{P_2} = GL_1 \times GL_2, \qquad L_G = GL_3,$$

and

$$A_{P_0} = \mathbb{G}_m^3 = \operatorname{diag}(\lambda, \mu, \nu), \qquad A_{P_1} = \mathbb{G}_m^2 = \operatorname{diag}(\lambda, \lambda, \mu),$$
$$A_{P_2} = \mathbb{G}_m^2 = \operatorname{diag}(\lambda, \mu, \mu), \qquad A_G = \mathbb{G}_m = \operatorname{diag}(\lambda, \lambda, \lambda).$$

2 April 1: real groups, symmetric spaces, boundaries.

Last time, we were working with a connected, not necessarily reductive, linear algebraic group $G/\mathbb{Q}$ with unipotent radical N_G and reductive quotient $L_G = G/N_G$. We chose $\tilde{L} \subset G$ such that $\tilde{L} \xrightarrow{\sim} L_G$ (unique up to conjugation by N_G). We let A_G be the maximal split torus in $Z(L_G)$ and $R_S(G)$ the preimage of A_G in G. We let $M_G = \bigcap_{\chi: G \to \mathbb{G}_m/\mathbb{Q}} \ker(\chi^2)$. This is "almost the complement" of A_G; it may be disconnected. For example, if $G = GL_2$, then $A_G = \mathbb{G}_m$ and $M_G = \{g \in GL_2 \mid \det g = \pm 1\}$, which is indeed disconnected.

2.1 Subgroups over $\mathbb{R}$

Even though G is connected as an algebraic group, $G(\mathbb{R})$ might not be connected. However, we can say that $\pi_0(G(\mathbb{R}))$ is a finite abelian group of exponent 2. For example, if $G = GL_2$, then $\pi_0(GL_2(\mathbb{R})) \cong \{\pm 1\}$ (the components of positive and negative determinant). Let $G(\mathbb{R})^+$ be the connected component of 1 in $G(\mathbb{R})$. This is not an algebraic group; for example $GL_2(\mathbb{R})^+ = \{g \in GL_2(\mathbb{R}) \mid \det g > 0\}$ is not the real points of any algebraic group.

Every compact subgroup of $G(\mathbb{R})$ is contained in a maximal one. Any two maximal compact subgroups are conjugate by $G(\mathbb{R})^+$. If $U_\infty \subset G(\mathbb{R})$ is a maximal compact subgroup, then $\pi_0(U_\infty) \xrightarrow{\sim} \pi_0(G(\mathbb{R}))$. Since U_∞ must have trivial intersection with the unipotent radical, we have an injection $U_\infty \hookrightarrow L_G(\mathbb{R})$ where the image is again maximal compact.

If $U_\infty \subset G(\mathbb{R})$ is maximal compact and $P \subset G$ is parabolic, then we have the Iwasawa decomposition

$$U_\infty P(\mathbb{R}) = P(\mathbb{R}) U_\infty = G(\mathbb{R}),$$

and $U_\infty \cap P(\mathbb{R})$ is maximal compact in $P(\mathbb{R})$. (Note that this is better than in p-adic land, where these aren't true unless the maximal compact is "well-positioned" with respect to the parabolic.)

Any maximal compact is contained in a Levi subgroup $\tilde{L}$, not necessarily unique.

Example 2.1.1. If $G = SL_2 \ltimes M_{2\times 2}$, where $M_{2\times 2}$ is the additive group of 2×2 matrices, with the semidirect product action of SL_2 given by

$$(g, 0)(1, x)(g, 0)^{-1} = (1, (g^{-1})^T x g^{-1}),$$

we can take $U_\infty = SO(2) \times \{0\}$ (where SO_2 is $\{g \in SL_2(\mathbb{R}) \mid g^T g = \mathrm{id}_2\}$). Then $U_\infty \subset SL_2 \times \{0\}$, which is a Levi, but also

$$U_\infty \subset \{(g, 1 - g^T g) \mid g \in SL_2\} = (\mathrm{id}_2, -\mathrm{id}_2)(SL_2 \times \{0\})(\mathrm{id}_2, -\mathrm{id}_2)^{-1},$$

which is another Levi. (The containment of U_∞ is because $1 - g^T g = 0$ if $g \in SO(2)$.)

Definition 2.1.2. If $U_\infty \subset \tilde{L}$ is a maximal compact in a Levi, then there is a unique $\theta = \theta_{\tilde{L}, U_\infty} : \tilde{L} \to \tilde{L}$ which is an automorphism over $\mathbb{R}$ satisfying $\theta^2 = 1$ such that

$$U_\infty = \{g \in \tilde{L}(\mathbb{R}) \mid \theta(g) = g\}.$$

This is the Cartan involution.

If $P \subset G$ is a parabolic, then P has a unique Levi $\tilde{L}_{P,\theta}$ stable under θ. Concretely,

$$\tilde{L}_{P,\theta} = \tilde{L} \cap P \cap \theta(\tilde{L} \cap P).$$

These behave well under conjugation by $G(\mathbb{R})$. For example, if $g \in G(\mathbb{R})$,

$$\theta_{g\tilde{L}g^{-1}, gU_\infty g^{-1}} = \mathrm{conj}_g \circ \theta \circ \mathrm{conj}_{g^{-1}}$$

and various other compatibilities could be written out as well.

Definition 2.1.3. By a lifting $\tilde{A}$ of A_G, we mean some $\tilde{A} \subset G$ with $\tilde{A} \xrightarrow{\sim} A_G$. (Again such a lifting might be contained in more than one Levi component.) By an essentially maximal compact subgroup $\tilde{U}_\infty \subset G(\mathbb{R})$, we mean a subgroup of the form $\tilde{U}_\infty = \tilde{A}(\mathbb{R})U_\infty$ where $U_\infty \subset G(\mathbb{R})$ is a maximal compact and $\tilde{A}(\mathbb{R}) \subset G$ is a lifting of A_G normalized by U_∞. (Richard made up this word.)

Choosing $\tilde{U}_\infty$ is the same thing as choosing the components U_∞ and $\tilde{A}$: U_∞ is the unique maximal compact in $\tilde{U}_\infty$, and $\tilde{A}(\mathbb{R}) = \tilde{U}_\infty \cap R_S(G)(\mathbb{R})$. We have $U_\infty \supset \tilde{A}(\mathbb{R})^{tor}$: since $\tilde{A}$ splits, we have $\tilde{A}(\mathbb{R}) \cong (\mathbb{R}^\times)^d$), and $\tilde{A}(\mathbb{R})^{tor}$ is finite, so if U_∞ didn't contain it you could just multiply it in and get a bigger compact subgroup. Therefore $\tilde{U}_\infty = \tilde{A}(\mathbb{R})^+ U_\infty$ (and we have $\tilde{A}(\mathbb{R})^+ \cong (\mathbb{R}_{>0})^d$).

Any maximal compact is contained in some essentially maximal compact. Any lifting $\tilde{A}$ of A_G is as well. Any two essentially maximal compacts are conjugate by $G(\mathbb{R})^+$. Any essentially maximal compact embeds into $L_G(\mathbb{R})$ with image again an essentially maximal compact subgroup. Any essentially maximal compact is contained in $\tilde{L}(\mathbb{R})$ for some Levi component $\tilde{L} \xrightarrow{\sim} L_G$, which is not necessarily unique.

2.2 Symmetric spaces

We will use the definition of Borel and Serre.

Definition 2.2.1. By a symmetric space for G we will mean a pair $(X, \{\tilde{L}_x\}_{x \in X})$ where X is a smooth manifold with a smooth transitive action of $G(\mathbb{R})$, and for all $x \in X$, $\tilde{L}_x \subset G$ is a choice of Levi subgroup defined over $\mathbb{R}$ such that

1. $\tilde{L}_{gx} = g\tilde{L}_x g^{-1}$ and
2. $\mathrm{Stab}_{G(\mathbb{R})}(x) =: \tilde{U}_x$ is an essentially maximal compact subgroup of $\tilde{L}_x(\mathbb{R})$.

We will write U_x for the maximal compact in $\tilde{U}_x$.

(If G is reductive, then it has a unique Levi component G, and $\tilde{L}_x = G$ for all $x \in X$, so we wouldn't need to keep track of this data.)

These exist. Choose $\tilde{L} \subset G$ a Levi component and $\tilde{U}_\infty \subset \tilde{L}(\mathbb{R})$ an essentially maximal compact subgroup. We can define $X = G(\mathbb{R})/\tilde{U}_\infty$ and $\tilde{L}_{g\tilde{U}_\infty} = g\tilde{L}g^{-1}$. In fact, any two such spaces are isomorphic, but not necessarily *uniquely* isomorphic, which is why we're defining this at all. Why? Suppose we have $(X', \{\tilde{L}'_x\})$. Then we can find some $x' \in X'$ with $\tilde{L}'_{x'} = \tilde{L}$, and then some $h \in \tilde{L}(\mathbb{R})$ such that

$$\mathrm{Stab}_{G(\mathbb{R})}(hx') = h\,\mathrm{Stab}_{G(\mathbb{R})}(x')h^{-1} = \tilde{U}_\infty,$$

and then the map $X \to X'$, $g\tilde{U}_\infty \mapsto g(hx')$, is an isomorphism. But our choices of x', h might not be unique.

(Note: A_G is maximal split over $\mathbb{Q}$, possibly not maximal over bigger fields.)

Suppose $N \subset N_G$ is normal in G. Then $(N(\mathbb{R})\backslash X, \{\tilde{L}_x\})$ is a symmetric space for G/N. $X \to N(\mathbb{R})\backslash X$ is an $N(\mathbb{R})$-torsor. It turns out that it often suffices to study $N(\mathbb{R})\backslash X$ as N varies over some filtration of N_G with abelian graded pieces.

Fix $P \subset G$ a parabolic and $(X, \{\tilde{L}_x\})$ a symmetric space for G. For $x \in X$, let

$$\tilde{L}_{P,x} := L_{P,\theta_{U_x}},$$

the unique Levi component for P in $\tilde{L}_x$ fixed by θ_{U_x}. We have

$$\tilde{L}_{gPg^{-1},gx} = g\tilde{L}_{P,x}g^{-1}.$$

Definition 2.2.2. Let $z \in Z(L_P)(\mathbb{R})$. It lifts to a unique $\tilde{z}_x \in \tilde{L}_{P,x}$. Define the "dot action"

$$z._P x = z.x = \tilde{z}_x x.$$

This gives a smooth action of $Z(L_P)(\mathbb{R})$ on X which commutes with the action of $P(\mathbb{R})$ on X: if $g \in P(\mathbb{R})$, $\tilde{z}_{gx}$ is the lifting of z to $\tilde{L}_{gx,P}$; since $g \in P$, $gPg^{-1} = P$, so this is the same as $g\tilde{L}_{x,P}g^{-1}$; so $\tilde{z}_{gx} = g\tilde{z}_x g^{-1}$. Therefore

$$z.(gx) = g\tilde{z}_x g^{-1}gx = g(z.x).$$

Furthermore, the action of $A_P(\mathbb{R})$ (resp. $A_P(\mathbb{R})^+$) factors through $(A_P/A_G)(\mathbb{R})$ (resp. $(A_P/A_G)(\mathbb{R})^+$), because by definition of the symmetric space, A_G acts trivially on X. The action of $A_P(\mathbb{R})^+ \times M_P(\mathbb{R})$ on X (using the dot action of $A_P(\mathbb{R})^+$) is transitive, and the stabilizer of $x \in X$ is

$$A_G(\mathbb{R}) \times (U_x \cap M_P(\mathbb{R})).$$

The advantage of splitting up the action like this is that the stabilizers of points in the first part are now all the same.

2.3 Example for GL_2

Example 2.3.1. Let $G = GL_2$ and $X = \mathbb{H} = \{\tau = x + iy \in \mathbb{C} \mid y > 0\}$. As usual, if $ad - bc > 0$, the action is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \tau \mapsto \frac{a\tau + b}{c\tau + d}.$$

The action of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is set to fix i. We have $\tilde{L}_\tau = GL_2$ for all τ. We have

$$\tilde{U}_i = GO(2) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a^2 + b^2 \neq 0 \right\}$$

containing

$$U_i = O(2) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a^2 + b^2 = 1 \right\} \cdot \left\{ 1, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

We have $A_G(\mathbb{R}) = \mathbb{R}^\times$, and it is central.

Let $B \subset GL_2$ be the upper triangular parabolic subgroup and $N_B \subset B$ the strictly upper triangular unipotent group. We have $A_B = L_B = B/N_B \cong \mathbb{G}_m^2$. We have

$$M_B = \{\pm 1\}^2 \ltimes \mathbb{G}_a = \left\{ \begin{pmatrix} \pm 1 & * \\ 0 & \pm 1 \end{pmatrix} \right\}.$$

The Cartan involution θ_{U_i} is $g \mapsto (g^{-1})^T$, so $\tilde{L}_{B,i} = T \cong \mathbb{G}_m^2$ is the diagonal torus. Then

$$\tilde{L}_{B,x+iy} = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} T \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}^{-1} = \left\{ \begin{pmatrix} \alpha & x(\beta - \alpha) \\ 0 & \beta \end{pmatrix} \mid \alpha, \beta \in \mathbb{G}_m \right\}$$

since $\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} i = x + iy$. Write $(\alpha, \beta) \in L_B(\mathbb{R})^+$ for $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$. Then

$$(\alpha, \beta).\tau = x + \frac{\alpha}{\beta} yi$$

(in contrast to the usual action of (α, β), which would multiply all of $x + yi$ by α/β instead of leaving x alone) because if $\tau = x + iy$, we have

$$(\alpha, \beta).\tau = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} ((\alpha, \beta).i) = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} ((\alpha/\beta)i) = x + \frac{\alpha}{\beta} yi.$$

2.4 Boundaries

Consider $(A_P(\mathbb{R})^+\backslash X, \{\tilde{L}_{x,P}\})$. This is a symmetric space for P which we will denote by X_P. It is canonical once X is fixed. The map $X \to X_P$ is an $(A_P/A_G)(\mathbb{R})^+$-fibration. For example, $\mathbb{H} \to \mathbb{R}$, $x+iy \mapsto x$, is an $\mathbb{R}^\times_{>0}$-fibration. If $g \in G(\mathbb{R})$, the action $g : X \to X$ induces an action $g : X_P \to X_{gPg^{-1}}$ commuting with $X \to X_P$ and $X \to X_{gPg^{-1}}$.

A problem with these locally symmetric spaces is that quotienting them by natural discrete subgroups produces noncompact spaces. We would like to extend them by some boundaries to make the quotients compact. The idea is that $(A_P/A_G)(\mathbb{R})^+ \cong (\mathbb{R}^\times_{>0})^d \subset (\mathbb{R}^\times_{\geq 0})^d$, so we want to extend $X \to X_P$ to a fibration of the slightly bigger group $(\mathbb{R}^\times_{\geq 0})^d$ compatibly as we vary P.

A_P/A_G acts on N_P/N_G. Just like with maximal tori, we can find a basis $\Delta(P)$ of the weights of A_P/A_G on N_P/N_G (meaning that any weight of A_P/A_G on N_P/N_G is uniquely a $\mathbb{Z}_{\geq 0}$-linear combination of the elements of $\Delta(P)$). We have

$$\prod_{\alpha \in \Delta(P)} \alpha : A_P/A_G \xrightarrow{\sim} \mathbb{G}_m^{\Delta(P)}.$$

For $Q \supset P$, we have a natural map $A_Q \hookrightarrow A_P$, giving in the other direction a map $\Delta(P) \to \Delta(Q) \coprod \{\text{triv}\}$. Define

$$X(P) = X \times_{A_P(\mathbb{R})^+} \overline{(A_P/A_G)(\mathbb{R})^+}$$

where by the second term we mean the closure of $(A_P/A_G)(\mathbb{R})^+ \cong (\mathbb{R}^\times_{>0})^{\Delta(P)}$ in $\mathbb{R}^{\Delta(P)}$, that is, it looks like $(\mathbb{R}^\times_{\geq 0})^{\Delta(P)}$, and by the relative product we mean the direct product modded out by the action $a(x,b) = (a.x, ab)$ of $A_P(\mathbb{R})^+$. We have a map $X \hookrightarrow X(P)$, $x \mapsto (x, 1)$, which is in fact an open embedding, because the image of $A_P(\mathbb{R})^+$ is open in $\overline{(A_P/A_G)(\mathbb{R})^+}$.

We have a map $X(P) \to X_P$ by projecting onto the first factor X and composing with $X \to X_P$, but also $X(P)$ contains $e(P) := X \times_{A_P(\mathbb{R})^+} \{0\}$, which projects isomorphically onto X_P because $A_P(\mathbb{R})^+$ acts trivially on $\{0\} \subset \overline{(A_P/A_G)(\mathbb{R})^+}$.

Example 2.4.1. Look at GL_2 and $\mathbb{H}$ again. We have

$$X(B) = \mathbb{H} \times_{\mathbb{R}^\times_{>0}} \mathbb{R}_{\geq 0}$$

where for $a \in \mathbb{R}^\times_{>0}$, $a(x+iy, b) = (x+iay, ab)$. So $X(B) = \mathbb{H} \coprod \mathbb{R}$ as sets. We are "gluing an extra copy of $\mathbb{R}$ (corresponding to $e(B)$) at $i\infty$".

If $g \in G(\mathbb{R})$, $g : X \to X$ commutes with the containments of X in $X(P)$ and $X(gPg^{-1})$ and the map $g : X(P) \to X(gPg^{-1})$, and so does $g : e(P) \to e(gPg^{-1})$.

If $Q \supset P$, the map $(A_Q/A_G)(\mathbb{R})^{(+)} \to (A_P/A_G)(\mathbb{R})^{(+)}$ induces

$$\overline{(A_Q/A_G)(\mathbb{R})^+} \hookrightarrow \overline{(A_P/A_G)(\mathbb{R})^+}$$

which can also be written

$$\begin{aligned} (\mathbb{R}^\times_{\geq 0})^{\Delta(Q)} &\to (\mathbb{R}^\times_{\geq 0})^{\Delta(P)} \\ (t_\alpha) &\mapsto \begin{pmatrix} t_{\beta|_{A_Q}} & \text{if} & \beta|_{A_Q} \in \Delta(Q) \\ 1 & \text{if} & \beta|_{A_Q} = \text{triv} \end{pmatrix}. \end{aligned}$$

This gives an open embedding $X(Q) \hookrightarrow X(P)$ commuting with $X(Q) \to X_Q$, $X(P) \to X_P$, and the map $X_Q \to X_P$ which is an $(A_P/A_Q)(\mathbb{R})^+$-torsor coming from the fact that $X_P \cong (X_Q)_P$. We have

$$X(P) = \coprod_{Q \supset P} e(Q)$$

as sets, and in particular $e(G) = X$. Then if $\overline{e(Q)}$ is the closure of $e(Q)$ in $X(P)$, we have

$$\overline{e(Q)} = \coprod_{Q \supset R \supset P} e(R).$$

Next week we will glue these as parabolics vary without necessarily being contained in each other.

3 April 6: Borel-Serre compactification.

3.1 Recap and SL_3 example

We have a connected linear algebraic group G acting transitively on X (with stabilizers having a particular structure described earlier), and for a parabolic $P \subset G$, the group $(A_P/A_G)(\mathbb{R})^+ \cong (\mathbb{R}^\times_{>0})^?$ has a dot action on X which commutes with the action of P. (It is not the restricted action from P, because the group is not naturally a subgroup of P, only its quotient by the unipotent radical.) We defined

$$X_P := A_P(\mathbb{R})^+ \backslash X$$

and saw that it is naturally a symmetric space for P. $X \to X_P$ is a fibration with fiber

$$(A_P/A_G)(\mathbb{R})^+ \cong (\mathbb{R}^\times_{>0})^{\Delta(P)} \subset (\mathbb{R}^\times_{\geq 0})^{\Delta(P)} \cong \overline{(A_P/A_G)(\mathbb{R})^+}.$$

We defined the corner associated to P by

$$X(P) = X \times_{A_P(\mathbb{R})^+} \overline{(A_P/A_G)(\mathbb{R})^+}.$$

It contains X as the dense open subset of points where no coordinates of $\overline{(A_P/A_G)(\mathbb{R})^+}$ are 0 and thus the product doesn't do anything; the complement is where some coordinates are 0, thus fixed by $A_P(\mathbb{R})^+$, and you get quotients of X. It still maps down to X_P but is a fibration for $(\mathbb{R}^\times_{\geq 0})^{\Delta(P)}$ instead of $(\mathbb{R}^\times_{>0})^{\Delta(P)}$. It also contains the subset $e(P)$ of points where all the coordinates in the second factor are 0, which maps isomorphically to X_P.

We saw that when $X = \mathbb{H}$ and P is the upper triangular Borel, $X(P)$ is $\mathbb{H} \coprod \mathbb{R}$ where $\mathbb{R}$ is glued in at $i\infty$.

For $g \in G(\mathbb{R})$, we have $g : X(P) \to X(gPg^{-1})$, taking $e(P)$ to $e(gPg^{-1})$. If $Q \supset P$, we have an open embedding $X(Q) \hookrightarrow X(P)$, and in fact

$$X(P) = \coprod_{Q \supset P} e(Q)$$

$$\overline{e(Q)} = \coprod_{Q \supset R \supset P} e(R)$$

where by $\overline{e(Q)}$ we mean the closure in $X(P)$. Note that these disjoint unions are finite.

Example 3.1.1. For SL_3,

$$B = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\}, \qquad A_B = \mathbb{G}_m^2$$

is contained in two maximal parabolics

$$P_1 = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \right\}, \qquad A_{P_1} = \mathbb{G}_m,$$

$$P_2 = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\}, \qquad A_{P_2} = \mathbb{G}_m.$$

The following is a symmetric space:

$$X = \{x \in M_{3\times 3}(\mathbb{R}) \mid x^T = x, \det(x) = 1, x > 0\}$$

with the action of $g \in SL_3(\mathbb{R})$ given by $g(x) = gxg^T$ (since SL_3 is reductive we don't need to keep track of Levi components). It has 5 real dimensions (6 minus 1 for the determinant condition). What is $X(P_1)$? We have a map $X_{P_1} \to X_{GL_2}$ by modding out by the action of N_{P_1}, because GL_2 is the Levi component of P_1 via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & (ad-bc)^{-1} \end{pmatrix}.$$

The fibers of $X_{P_1} \to X_{GL_2}$ are $\mathbb{R}^2$ (i.e. $N_{P_1}(\mathbb{R})$). Since $X_{GL_2} = \mathbb{H}$, X_{P_1} is 4-dimensional. Then we have $X(P_1) \supset X$ mapping to X_{P_1} with fibers $\mathbb{R}^\times_{\geq 0}$ and $\mathbb{R}^\times_{>0}$ respectively, and

$$X(P_1) = X \coprod (X_{P_1} = e(P_1)).$$

Next, $X(B)$ contains $X(P_1)$ and $X(P_2)$. The Levi of B is $\mathbb{G}_m^2$, so we get a fibration $X_B \to X_{\mathbb{G}_m^2}$ with fibers $N(\mathbb{R})$, where

$$N = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

The fibers $N(\mathbb{R})$ are not abelian, but we can factor the fibration into the abelian ones

$$X_B \xrightarrow{\mathbb{R}} X_B / \left\{ \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \xrightarrow{\mathbb{R}^2} X_{\mathbb{G}_m^2} = pt$$

(here $\left\{ \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$ is the center of the unipotent radical N), and we see that X_B has 3 real dimensions. Then we have $X(B) \supset X$ mapping to X_B with fibers $(\mathbb{R}^\times_{\geq 0})^2$ and $(\mathbb{R}^\times_{>0})^2$. $X(P_1)$ inside $X(B)$ fibers over X_B with fibers $\mathbb{R}^\times_{>0} \times \mathbb{R}^\times_{\geq 0}$ and $X(P_2)$ is the same with $\mathbb{R}^\times_{\geq 0} \times \mathbb{R}^\times_{>0}$.

3.2 Borel-Serre spaces

Definition 3.2.1. Let

$$X^{BS} = \varinjlim_{\substack{P\subset G \\ \text{rational parabolic}}} X(P) = \left(\bigcup_{\substack{P\subset G \\ \text{rational parabolic}}} X(P) \right) / \sim$$

where if $Q \supset P$ then $X(Q)$ is identified with its image in $X(P)$. (Keep in mind that unless P is minimal, it contains infinitely many smaller parabolics, though it is contained in only finitely many ones.)

Example 3.2.2. For GL_2, we get (as a set) $X^{BS}_{GL_2} = \mathbb{H} \coprod \left(\coprod_{\mathbb{P}^1(\mathbb{Q})} \mathbb{R} \right)$: we glue in a $\mathbb{R}$ for each rational point on the real line and also $i\infty$.

We have

$$X^{BS} = \coprod_P (e(P) \cong X_P)$$

containing $X = e(G)$ as an open dense subset. Also

$$X^{BS}_P \cong \overline{e(P)} = \coprod_{Q\subset P} e(Q).$$

Also $X(P) \cap X(Q) = X(R)$ where R is the minimal parabolic containing P and Q (there always exists such a parabolic, e.g. G).

Fact: X^{BS} is Hausdorff, contractible, and a countable union of compact subsets. $G(\mathbb{Q})$ acts continuously on X^{BS}, with γ taking $X(P)$ to $X(\gamma P \gamma^{-1})$ and $e(P) \to e(\gamma P \gamma^{-1})$. (We are having a hard time deciding whether we should call $X(P)$ or $e(P)$ the "corner". We could say that the latter is the "small corner" and the former is the "big (thickened) corner".) Note that $G(\mathbb{R})$ does *not* act on X^{BS}, since non-rational points would have to take rational parabolics to non-rational parabolics.

Another thing we should note is that $X(P) \xrightarrow{(\mathbb{R}^\times_{>0})^{\Delta(P)}} X_P$ nearly has a product structure. Fix $x \in X$ going to $x_P \in X_P$. We had the map

$$\begin{aligned} (A_P/A_G)(\mathbb{R})^+ \times M_P(\mathbb{R}) \times X &\to X \\ (a, h) &\mapsto h(a^{-1}.x). \end{aligned}$$

But we also have

$$\begin{aligned} (A_P/A_G)(\mathbb{R})^+ \times M_P(\mathbb{R}) &\to (A_P/A_G)(\mathbb{R})^+ \times X_P \\ (a, h) &\mapsto (a, hx_P). \end{aligned}$$

In fact the first map factors through the second and gives an isomorphism

$$(A_P/A_G)(\mathbb{R})^+ \times X_P \xrightarrow{\sim} X.$$

This isomorphism extends to

$$\mu_{P,x} = \mu_x : \overline{(A_P/A_G)(\mathbb{R})^+} \times X_P \xrightarrow{\sim} X(P)$$

and further to

$$\mu_x : \overline{(A_P/A_G)(\mathbb{R})^+} \times X_P^{BS} \xrightarrow{\sim} \bigcup_{Q \subset P} X(Q).$$

Example 3.2.3. If $G = GL_2$ and $P = B$, we have

$$\begin{aligned} \mu_i : \mathbb{R}_{>0}^\times \times \mathbb{R} &\xrightarrow{\sim} \mathbb{H} \\ (a, x) &\mapsto x + a^{-1} i. \end{aligned}$$

How does this depend on the choice of base point x? If $y \in X$, we have $y = k(\nu_{P,x}(y)^{-1}.x)$ for some $k \in M_P(\mathbb{R})$ and $\nu_{P,x}(y) \in A_P(\mathbb{R})^+$. Then if $a \in (A_P/A_G)(\mathbb{R})^+$ and $z \in X_P$, we have

$$\mu_{P,y}(a, z) = \mu_{P,x}(\nu_{P,x}(y)a, z).$$

This is a very mild dependence! It only rescales the first coordinate by a fixed element. Also,

$$\mu_{\gamma P \gamma^{-1}, \gamma x}(\gamma a, \gamma z) = \gamma \mu_{P,x}(a, z).$$

We can use these maps to define neighborhoods of ∞. If $\underline{t} \in \mathbb{R}_{>0}^{\Delta(P)}$, let

$$X(P)_{\leq \underline{t}, x} = \mu_x \left(\prod_{\alpha \in \Delta(P)} [0, t\alpha] \times X_P \right)$$

(a compact neighborhood box of the corner). Then

$$\overline{X(P)_{\leq \underline{t}, x}} = \mu_x \left(\prod_{\alpha \in \Delta(P)} [0, t\alpha] \times X_P^{BS} \right)$$

is a compact neighborhood of $\overline{e(P)}$. "A neighborhood of a boundary component is homeomorphic to that boundary component times a hypercube."

Example 3.2.4.

$$\mathbb{H}_{i, \leq t} = (\mathbb{R} \times i[1/t, \infty)) \coprod \mathbb{R} \cong \mathbb{R} \times [0, t].$$

So neighborhoods of the corner at infinity are horizontal strips near infinity.

For $\gamma \in G(\mathbb{Q})$,

$$\gamma X(P)_{x, \leq t} = X(\gamma P \gamma^{-1})_{x, \leq \underline{t}(\nu_{\gamma P \gamma^{-1}, x}(\gamma x))}.$$

3.3 Manifolds with corners

Right now, X^{BS} is a topological space. We would like it to have a manifold structure. X is a smooth manifold, but X^{BS} is not, and not even a manifold with boundary (then it would have to locally look like a half-space). What it actually is is a manifold with corners. The literature on manifolds with corners is "a mess"; Richard recommends D. Joyce, "On manifolds with corners" [9].

Definition 3.3.1. Let Y be a Hausdorff, paracompact topological space. A chart on Y is a homeomorphism $U \to \varphi(U) \subset Y$ where U is an open subset of $\mathbb{R}^n_{\geq 0}$. (U, φ) and (V, ψ) are compatible if

$$\psi^{-1}\varphi : \varphi^{-1}(\varphi U \cap \psi V) \to \psi^{-1}(\varphi U \cap \psi V)$$

and its inverse extend to smooth maps on some open neighborhoods of the source and target in $\mathbb{R}^n$. An atlas is a collection of pairwise compatible charts (U_i, φ_i) with $\bigcup \varphi_i U_i = Y$. Any atlas extends to a unique maximal atlas. An n-dimensional manifold with corners is the data of Y together with a maximal atlas.

We can define

- smooth maps (again, requiring extension to open subsets of $\mathbb{R}^n$).
- the tangent space and, at a boundary point, its "inward pointing cone" of tangent vectors pointing into the manifold.
- the depth k points $S^k(Y)$, which are those for which k coordinates are 0 in a chart (so interior points are depth 0, points on an edge but not a corner are depth 1, etc.) $S^k(Y)$ is a $(n-k)$-dimensional smooth manifold. We have $Y = \coprod S^k(Y)$ and
$$\overline{S^k(Y)} = \bigcup_{l \geq k} S^l(Y).$$
- $\partial Y = \{(y, \beta) \mid y \in Y \text{ and } \beta \in \varprojlim_{U \ni y} \pi_0(U \cap S^1(Y))\}$. For example the boundary of a simple 2d corner $\mathbb{R}^2_{\geq 0}$ is two disconnected rays, each closed off at a copy of the depth-2 point, because the depth-2 point disconnects the two incoming depth-1 lines. Similarly, the boundary of a "teardrop", where you take a 2d corner and join up the two boundary lines to form a disc with a corner, is one closed interval (in which again both endpoints of the boundary map to the same point in the teardrop). ∂Y is an $(n-1)$-dimensional manifold with corners.
- $\partial^k Y = \partial(\partial^{k-1} Y)$. For example, if Y is a simple 3d corner/wedge $\mathbb{R}^3_{\geq 0}$, ∂Y is three disjoint 2d corners (copies of $\mathbb{R}^2_{\geq 0}$, whose edge lines glue together in pairs in $\mathbb{R}^3_{\geq 0}$), and then ∂^2 is *six* disconnected rays, because it keeps track of in "which order" you include an edge line into the boundary components.

Definition 3.3.2. An $\langle N \rangle$-manifold is a manifold with corners Y such that we have a decomposition

$$\partial Y = \partial_1 Y \coprod \cdots \coprod \partial_N Y$$

as a topological space with $\partial_i Y \hookrightarrow Y$. For example, the teardrop is not an $\langle N \rangle$-manifold for any N, because the boundary is connected so we would have to have $N = 1$, but ∂ does not inject into the teardrop.

Definition 3.3.3. If a manifold with corners has the structure of an $\langle N \rangle$-manifold, we say Y is a *manifold with embedded corners.* For example, our simple 2d/3d corners/wedges are manifolds with embedded corners. A $\langle 0 \rangle$-manifold is a manifold, and a $\langle 1 \rangle$-manifold is a manifold with boundary.

Theorem 3.3.4 (see paper of Johnson for reference)**.** *If Y is a compact $\langle N \rangle$-manifold, then it has a triangulation (i.e. a homeomorphism to a finite union of simplices joined along their boundaries to form a simplicial complex) such that for all $I \supset \{1, \ldots, N\}$, $\bigcap_{i \in I} \partial_i Y$ corresponds to a closed subcomplex. "You can triangulate it compatibly with the boundary structure."*

(Richard suspects that this is actually true for all manifolds with corners, but could not find a reference.)

An important Borel-Serre result is that X^{BS} is an $\langle N \rangle$-manifold, where you can take N equal to the number of $G(\mathbb{Q})$-conjugacy classes of maximal parabolic subgroups. We have

$$\partial X^{BS} = \coprod_{[P]} \left(\coprod_{P' \in [P]} X^{BS}_{P'} \right)$$

where $[P]$ runs over $G(\mathbb{Q})$-conjugacy classes of maximal parabolics, P' runs over parabolics in $[P]$, and $X^{BS}_{P'} = \overline{e(P')}$.

3.4 Locally symmetric spaces

Definition 3.4.1. We call a subgroup $\Gamma \subset G(\mathbb{Q})$ arithmetic if it is commensurable with $\rho^{-1} GL_M(\mathbb{Z})$ for some $\rho : G \hookrightarrow GL_M$, meaning that $\Gamma \cap \rho^{-1} GL_M(\mathbb{Z})$ is finite-index in both Γ and $\rho^{-1} GL_M(\mathbb{Z})$.

Theorem 3.4.2 (Borel-Serre)**.** *If $\Gamma \subset G(\mathbb{Q})$ is arithmetic, then Γ acts properly on X^{BS}, meaning that*

$$\#\{\gamma \in \Gamma \mid \gamma\Omega \cap \Omega \neq \emptyset\} < \infty$$

for all $\Omega \subset X^{BS}$ compact, and $\Gamma \backslash X^{BS}$ is a compact Hausdorff $\langle N \rangle$-manifold (where N is the number of $G(\mathbb{Q})$-conjugacy classes of maximal parabolics).

If Γ is torsion-free, then $\pi : X^{BS} \to \Gamma \backslash X^{BS}$ is a covering map, i.e. for all $\overline{x} \in \Gamma \backslash X^{BS}$, there is $\overline{x} \in U \subset \Gamma \backslash X^{BS}$ open such that $\pi^{-1} U = \coprod U_i$, where $\pi : U_i \xrightarrow{\sim} U$, and Γ acts simply transitively on the U_i. ("π is topologically a Γ-torsor.")

We have

$$\partial \Gamma \backslash X^{BS} = \coprod_{[P]_{G(\mathbb{Q})}} \coprod_{[P']_\Gamma \subset [P]_{G(\mathbb{Q})}} (\Gamma \cap P'(\mathbb{Q})) \backslash X^{BS}_{P'}$$

where $[P]_{G(\mathbb{Q})}$ runs over $G(\mathbb{Q})$-conjugacy classes of maximal rational parabolics (a finite union, because any parabolic is conjugate to one containing any given minimal parabolic, which is contained in only finitely many parabolics) and $[P']_\Gamma$ runs over Γ-conjugacy classes in the $G(\mathbb{Q})$-conjugacy class (also a finite union). That is, the boundary of the locally symmetric space is a disjoint union of locally symmetric spaces for proper parabolics.

Iterating this, we conclude that $\Gamma\backslash X^{BS}$ has a finite triangulation compatible with the boundary in the sense described before.

Next time, we will talk about making this adelic, and then sheaves on these spaces.

4 April 8: adelic locally symmetric spaces.

Recall that we had a connected linear algebraic group $G/\mathbb{Q}$ with $G(\mathbb{R})$ acting on its symmetric space X, which is unique but not unique up to unique isomorphism. For each rational (defined over $\mathbb{Q}$) parabolic $P \subset G$, we defined a "partial compactification" $X(P)$ containing X as an open dense subset, and $X^{BS} = \bigcup_P X(P)/\sim$ (identifying the space for a parabolic as an open inside the space for any parabolic it contains), which again contains X as an open dense subset, and has an action of $G(\mathbb{Q})$ (but not $G(\mathbb{R})$). X^{BS} is an $\langle N\rangle$-manifold where N is the number of conjugacy classes of maximal parabolics, so that

$$\partial X^{BS} = \coprod_{\substack{[P] \\ G(\mathbb{Q})\text{-conjugacy classes} \\ \text{of max parabolics}}} \coprod_{P' \in [P]} X^{BS}_{P'}.$$

For $\Gamma \subset G(\mathbb{Q})$ an arithmetic subgroup, $\Gamma\backslash X \subset \Gamma\backslash X^{BS}$ is again a manifold with corners (or actually an $\langle N\rangle$-manifold). Γ acts properly on X^{BS} (a compact set meets only finitely many of its Γ-translates), so $\Gamma\backslash X$ is nice. It is compact; if Γ is torsion-free, $X^{BS} \to \Gamma\backslash X^{BS}$ is a covering map; the boundary of $\Gamma\backslash X^{BS}$ decomposes in a similar way as above—you can partition it into subsets indexed by conjugacy classes of maximal parabolics, giving the $\langle N\rangle$-manifold structure.

4.1 In adelic language

Open compact subgroups of $G(\mathbb{A}_f)$ may have torsion, which, as with arithmetic subgroups of $G(\mathbb{Q})$, we may wish to avoid. Let

$$\mathbb{A}^S = \prod_{v\notin S}{}' \mathbb{Q}_v.$$

Definition 4.1.1. We call $g \in GL_n(\mathbb{A}^S)$ neat if

$$\bigcap_{v\notin S} \left(\{\text{the subgroup of } \overline{\mathbb{Q}}_v^\times \text{ generated by eigenvalues of } g\} \cap \overline{\mathbb{Q}}^\times\right)^{tor} = \{1\}$$

where we note that since g is defined over $\mathbb{Q}_v$, if something is an eigenvalue of g, then all its Galois conjugates are. (So the idea is that we want to exclude two eigenvalues of g differing by a root of unity, but it's okay if e.g. you have cube roots of unity in one place and square roots of unity in another.)

We call $g \in G(\mathbb{A}^S)$ neat if there is $\rho : G \hookrightarrow GL_N$ with $\rho(g)$ neat; this is true if and only if for all $\rho : G \to GL_N$, $\rho(g)$ is neat.

Facts: g being neat depends only on its conjugacy class $[g]$, since it's a condition on the eigenvalues. If $g \in G(\mathbb{A}^S)$ is neat and $g_v \in G(\mathbb{Q}_v)$ for some $v \in S$, then $(g, g_v) \in G(\mathbb{A}^{S-\{v\}})$ is neat. (So once you're neat at a set of places, you can add whatever you want at other places.) If $\varphi : G \to H$ and $g \in G(\mathbb{A}^S)$ is neat, then $\varphi(g)$ is neat.

$U \subset G(\mathbb{A}^S)$ is called neat if all its elements are neat. If $U \subset G(\mathbb{A}^S)$ is an open compact subgroup then there is a neat open subgroup $V \subset U$ of finite index. If $U \subset G(\mathbb{A}^S)$ is neat then $U \cap G(\mathbb{Q})$ is torsion-free. (This is the point—to produce torsion-free arithmetic groups.)

If $U \subset G(\mathbb{A}^S)$ is an open compact subgroup, we can define a locally symmetric space

$$X_U = G(\mathbb{Q})\backslash(G(\mathbb{A}^\infty)/U \times X).$$

We can decompose

$$G(\mathbb{A}^\infty) = \coprod_i G(\mathbb{Q}) g_i U$$

where this disjoint union is finite (because of the finiteness of class groups and strong approximation for simply connected semisimple groups), so

$$X_U = \coprod_i (G(\mathbb{Q}) \cap g_i U g_i^{-1})\backslash X$$

where $G(\mathbb{Q}) \cap g_i U g_i^{-1}$ is an arithmetic subgroup of $G(\mathbb{Q})$ which is torsion-free if U is neat. Similarly, we can define

$$X_U^{BS} := G(\mathbb{Q})\backslash(G(\mathbb{A}^\infty)/U \times X^{BS}) = \coprod_i (G(\mathbb{Q}) \cap g_i U g_i^{-1})\backslash X^{BS}.$$

Then X_U^{BS} is a compact manifold with corners, in fact an $\langle N \rangle$-manifold. We have

$$\partial X^{BS} = \coprod_{\substack{[P] \\ G(\mathbb{Q})\text{-conjugacy classes} \\ \text{of max rat parabolics}}} \coprod_{g \in P(\mathbb{A}^\infty)\backslash G(\mathbb{A}^\infty)/U} X^{BS}_{P, gUg^{-1} \cap P(\mathbb{A}^\infty)}.$$

Here both disjoint unions are finite. In this decomposition, $(h, x) \in X^{BS}_{P, gUg^{-1} \cap P(\mathbb{A}^\infty)}$, where $h \in P(\mathbb{A}^\infty)$ and $x \in X_P^{BS}$, corresponds to $(hg, x) \in \partial X^{BS}$, where x is viewed as its image under the embedding $X_P^{BS} \to X^{BS}$.

We can furthermore compute the k-boundary:

$$\partial^k X_U^{BS} = \coprod_{[G \supset P_1 \supset \cdots \supset P_k]} \coprod_{g \in P_k(\mathbb{A}^\infty)\backslash G(\mathbb{A}^\infty)/U} X^{BS}_{P_k, gUg^{-1} \cap P_k(\mathbb{A}^\infty)}$$

where $[G \supset P_1 \supset \cdots \supset P_k]$ runs over $G(\mathbb{Q})$-conjugacy classes of chains of rational parabolics, each maximal in the previous one.

We will write $\partial_k X_U^{BS}$ for the image of $\partial^k X_U^{BS} \to X_U^{BS}$. We will write

$$\partial_k^\circ X_U^{BS} = \partial_k X_U^{BS} \setminus \partial_{k+1} X_U^{BS} = \coprod_{[P] : \dim A_P/A_G = k} \coprod_{g \in P(\mathbb{A}^\infty)\backslash G(\mathbb{A}^\infty)/U} X_{P, gUg^{-1} \cap P(\mathbb{A}^\infty)}$$

where $[P]$ runs over $G(\mathbb{Q})$-conjugacy classes of rational parabolics such that $\dim A_P/A_G = k$. We will denote the component of this associated to P by

$$\coprod_{g\in P(\mathbb{A}^\infty)\backslash G(\mathbb{A}^\infty)/U} X_{P,gUg^{-1}\cap P(\mathbb{A}^\infty)} =: \partial^\circ_{[P]} X_U^{BS}$$

and also write

$$\partial_{[P]} X_U^{BS} = \overline{\partial^\circ_{[P]} X_U^{BS}} = \bigcup_{g\in P(\mathbb{A}^\infty)\backslash G(\mathbb{A}^\infty)/U} X^{BS}_{P,gUg^{-1}\cap P(\mathbb{A}^\infty)}.$$

(which is no longer a disjoint union since taking BS might result in intersections in lower-dimensional strata.)

4.2 More topology of Borel-Serre spaces

Let $N \subset N_G$ be abelian and normal in G. Then

$$X_U^{BS} \to X^{BS}_{G/N,U \pmod{N(\mathbb{A}^\infty)}}$$

is a fibration whose fiber above $(g(U \pmod{N(\mathbb{A}^\infty)}), x)$ is

$$N(\mathbb{Q})\backslash N(\mathbb{A})/(gUg^{-1}\cap N(\mathbb{A}^\infty)) = (N(\mathbb{Q})\cap gUg^{-1})\backslash N(\mathbb{R})$$

since $N(\mathbb{Q})$ is dense in $N(\mathbb{A}^\infty)$. This is topologically a product of circles. That is, before quotienting by N we had a fibration by real vector spaces, and afterwards we have a fibration by quotients of real vector spaces, i.e. lattices. For example if $N = \mathbb{G}_a$, then we have $N(\mathbb{Q}) \cap$ (open compact subgroup of $N(\mathbb{A}^\infty))\backslash N(\mathbb{R})$, or $\mathbb{Q} \cap$ (open compact subgroup of $\mathbb{A}^\infty) \cap \mathbb{R}$, or $M\widehat{\mathbb{Z}}\backslash\mathbb{R}$ for some $M \in \mathbb{Q}^\times$, or $M\mathbb{Z}\backslash\mathbb{R}$.

(This picture doesn't look quite the same if N is not abelian. For example if we have the upper triangular unipotent for GL_3, we shouldn't take N to be all of it, but we could take N to be the subgroup that's only nonzero in the top right corner, which is abelian and normal in the big unipotent. So we would first get a fibration for the locally symmetric space for the Borel over the one for the quotient by this N, and then after quotienting the rest of the unipotent would be abelian, so we would get an iterated product of circle bundles.)

Assume U is neat. Then $G(\mathbb{Q})$ acts on $G(\mathbb{A}^\infty)/U \times X^{BS}$ properly and without fixed points (meaning each $\gamma \neq 1 \in G(\mathbb{Q})$ has no fixed points). Consequently, for all $g \in G(\mathbb{A}^\infty)$ and $x \in X^{BS}$, there is a compact neighborhood W of x, so $x \in W \subset X^{BS}$, such that if

$$\gamma(gU \times W) \cap (gU \times W) \neq \emptyset$$

for $\gamma \in G(\mathbb{Q})$, then $\gamma = 1$. That is, by properness we can find a neighborhood W such that this intersection is nontrivial only for finitely many γ; then we take the neighborhood and remove the γ-translates of it for those finitely many exceptional γ; since those γ don't fix (gU, x), we can still have a neighborhood afterwards. Then we see that $G(\mathbb{A}^\infty)/U \times X^{BS} \to X^{BS}$ is a covering map, because

$$\begin{aligned} W &\hookrightarrow X_U^{BS} \\ y &\mapsto [(g, y)] \end{aligned}$$

is an embedding and gives a neighborhood of $[(g,x)]$, and the preimage of this neighborhood in $G(\mathbb{A}^\infty)/U \times X^{BS}$ is the disjoint union

$$\coprod_{\gamma \in G(\mathbb{Q})} \gamma(gU \times W).$$

Now suppose $V \subset U$ is a closed subgroup (not necessarily open...e.g. just the identity). We can define

$$X_V^{BS} = G(\mathbb{Q}) \backslash \left(G(\mathbb{A}^\infty)/V \times X^{BS} \right)$$

(which is not necessarily a manifold with corners, and could be something very large, if V is not open). Let $\pi : X_V^{BS} \to X_U^{BS}$. Let $W \subset X_U^{BS}$ be a nice neighborhood of $[(g,x)]$ as above. Then we have a homeomorphism

$$\begin{aligned} U/V \times W &\xrightarrow{\sim} \pi^{-1}W \\ (u,y) &\mapsto (gu,y). \end{aligned}$$

Why is this injective? If $\gamma \in G(\mathbb{Q})$, and $\gamma(gu,y) = (gu',y')$, then the LHS is in $\gamma(gU \times W)$ and the RHS in $gU \times W$, so $\gamma = 1$, $y = y'$, and $uV = u'V$. Since $U/V \times W$ is compact and the map is a continuous injection, it is a homeomorphism onto its image. Also it is tautologically surjective.

From this we conclude that

$$X_V^{BS} \cong \varprojlim_{\substack{V \subset U' \subset U \\ U' \text{ open}}} X_{U'}^{BS}$$

homeomorphically, and in particular

$$X_{\{1\}}^{BS} = G(\mathbb{Q}) \backslash (G(\mathbb{A}^\infty) \times X^{BS}) = \varprojlim_U X_U^{BS}.$$

If $V \subset U$ is open with U neat, then $X_V \to X_U$ is a covering map of degree $[U : V]$. If V is normal in U, then $X_V \to X_U$ is Galois with group U/V. If $U, U', g \in G(\mathbb{A}^\infty)$ are such that $U' \supset g^{-1}Ug$, then we get a continuous map

$$\begin{aligned} g : X_U^{BS} &\to X_{U'}^{BS} \\ (h,x) &\mapsto (hg,x) \end{aligned}$$

for $h \in G(\mathbb{A}^\infty)$ and $x \in X^{BS}$ (which is in fact how U/V acts on X_V in the Galois case), which is again a covering map if U' is neat. These maps preserve the boundary structure: they take the boundary to the boundary, the preimage of the boundary is in the boundary, and stratifications are preserved, so $\partial_{[P]}^\circ X_U^{BS}$ is the preimage of $\partial_{[P]}^\circ X_{U'}^{BS}$.

Example 4.2.1. If $G(\mathbb{R})/A_G(\mathbb{R})^+$ is compact, then $X_U^{BS} = X_U$ is a finite set of points.

This is the setting we have worked in previously, but if we don't have a finite set of points, and really have a positive-dimensional manifold, how do we work with the automorphic forms? We could look at classical automorphic forms—real analytic functions or sections of a vector bundle on such a space—but it's very hard to do arithmetic with such things.

What is more arithmetic is the homology of these spaces. For example if $G(\mathbb{R})/A_G(\mathbb{R})^+$ is compact, the finite set of points just has an H^0, which are functions on the finite set, so in the general case looking at higher homologies seems like a reasonable generalization. But also we want the automorphic forms to have some weight, which we do by looking at the homology not just of the space itself but a locally constant sheaf on it.

4.3 Locally constant sheaves

Let F be a totally disconnected topological abelian group, meaning that for all $x_1 \neq x_2 \in F$, there are $U_1, U_2 \subset F$ open such that $x_i \in U_i$ and $U_1 \coprod U_2 = F$. For example $\mathbb{Z}_l, \mathbb{Q}_l, \overline{\mathbb{Q}}_l, \mathbb{Z}/l^2$.

Let $U \subset G(\mathbb{A}^\infty)$ be a neat open compact subgroup and $\rho : U \times F \to F$ a continuous representation. For example, we could have $G = GL_2$, $U = GL_2(\widehat{\mathbb{Z}})$, and $U \twoheadrightarrow GL_2(\mathbb{Z}_l)$ acting by the standard representation on $\mathbb{Z}_l^2$.

We will define a sheaf $\mathscr{F}_\rho$ on X_U^{BS} by, if $\pi : G(\mathbb{Q})\backslash(G(\mathbb{A}^\infty) \times X^{BS}) \to X_U^{BS}$ is the fibration by U we were talking abut before,

$$\mathscr{F}_\rho(W) = \{f : \pi^{-1}W \to F \mid f \text{ continuous}, f(xu) = \rho(u)^{-1}f(x) \forall u \in U, x \in \pi^{-1}W\}.$$

Lemma 4.3.1. *$\mathscr{F}_\rho$ is a locally constant sheaf.*

Proof. Let W be a sufficiently small open set in X_U^{BS}. We saw that $\pi^{-1}W \cong U \times W$. Then

$$\begin{aligned}\mathscr{F}_\rho(W) &= \{f : U \times W \to F \mid f \text{ continuous}, f(uv, x) = \rho(v)^{-1}f(u, x) \forall u, v \in U, x \in W\} \\ &\cong \{f : W \to F \mid f \text{ continuous}\}\end{aligned}$$

since the transformation condition means that $f(u, \cdot)$ is determined by $f(1, \cdot)$, and the continuity condition is unaffected because ρ is continuous; since F is totally disconnected, this is the same as

$$\{f : W \to F \mid f \text{ locally constant}\}$$

which is what a locally constant sheaf is. □

Lemma 4.3.2. *If $i : X_U \hookrightarrow X_U^{BS}$, then $\mathscr{F}_\rho \xrightarrow{\sim} Ri_*(\mathscr{F}_\rho|_{X_U})$ (so in particular the higher derived pushforwards are 0), and*

$$H^j(X_U^{BS}, \mathscr{F}_\rho) \xrightarrow{\sim} H^j(X_U, \mathscr{F}_\rho|_{X_U}).$$

Furthermore, we have

$$R\Gamma(X_U^{BS}, \mathscr{F}_\rho) \xrightarrow{\sim} R\Gamma(X_U, \mathscr{F}_\rho|_{X_U})$$

in $D(Ab)$.

Proof. This is true in the generality of locally constant sheaves on manifolds with corners. We need to compute $Ri_*(\mathscr{F}_\rho|_{X_U})$, which can be done locally. So we can assume that $\mathscr{F}_\rho$ is constant, and we can replace i by

$$\tilde{i} : \mathbb{R}^a_{>0} \times \mathbb{R}^b \hookrightarrow \mathbb{R}^a_{\geq 0} \times \mathbb{R}^b$$

near $\underline{0}$. Then if $\underline{A}$ is the constant sheaf with coefficients A, $R^j\tilde{i}_*\underline{A}$ is the sheaf associated to the presheaf

$$V \mapsto H^j(\tilde{i}^{-1}V, A)$$

where $\tilde{i}^{-1}V = V \cap (\mathbb{R}^a_{>0} \times \mathbb{R}^b)$, whose stalk at x is

$$\lim_{V \ni x} H^j(V \cap (\mathbb{R}^a_{>0} \times \mathbb{R}^b), A).$$

But for a suitable cofinal system of choices of V, $V \cap (\mathbb{R}^a_{>0} \times \mathbb{R}^b)$ is contractible (for example in the case of the 2d corner $\mathbb{R}^2_{\geq 0}$, with $a = 2$ and $b = 0$, all $\tilde{i}^{-1}$ does is remove the depth-2 point), so this is 0 if $j > 0$ and A if $j = 0$. □

Lemma 4.3.3. *Suppose Λ is a ring and F is a finitely generated Λ-module (same F as before—topological, totally disconnected, etc.), and U acts Λ-linearly on F. Then*

$$R\Gamma(X_U^{BS}, \mathscr{F}_\rho) \in D(\Lambda - Mod)$$

is represented by a bounded complex of finitely generated Λ-modules. If F is finite projective as a Λ-module, then $R\Gamma(X_U^{BS}, \mathscr{F}_\rho)$ is perfect, meaning that it is represented by a bounded complex of finite projective Λ-modules.

Proof. The important point is that X_U^{BS} has a finite triangulation and $\mathscr{F}_\rho$ is a locally constant sheaf of finitely generated Λ-modules (resp. finite projective Λ-modules). □

For the cohomology to be finitely generated you need Λ to be noetherian.

5 April 13: cohomology of sheaves on locally symmetric spaces.

5.1 Lemma from last time

Recall that we had a totally disconnected topological abelian group F and an open compact subgroup $U \subset G(\mathbb{A}^\infty)$, and a continuous action $\rho : U \times F \to F$. We defined a sheaf $\mathscr{F}_\rho$ on X_U^{BS} by

$$\mathscr{F}_\rho(W) = \{f : \pi^{-1}W \to F \mid f \text{ continuous}, f(xu) = u^{-1}f(x) \forall u \in U\}$$

where $\pi : G(\mathbb{Q})\backslash(G(\mathbb{A}^\infty) \times X^{BS}) \to X_U^{BS}$. We saw that $\mathscr{F}_\rho$ is a locally constant sheaf and that $R\Gamma(X_U^{BS}, \mathscr{F}_\rho) \in D^b(Ab)$ is the same as $R\Gamma(X_U, \mathscr{F}_\rho)$. Furthermore, if F is a Λ-module for some ring Λ and ρ is Λ-linear, then $R\Gamma(X_U^{BS}, \mathscr{F}_\rho) \in D^b(\Lambda - Mod)$ (which has a forgetful functor to $D^b(Ab)$). In particular, $H^i(X_U^{BS}, \mathscr{F}_\rho) = H^i(X_U, \mathscr{F}_\rho)$.

Last time, we stated the following lemma, which we will now sketch a proof of.

Lemma 5.1.1. *If F is a finitely generated Λ-module, then $R\Gamma(X_U^{BS}, \mathscr{F}_\rho)$ is represented by a bounded complex of finitely generated Λ-modules. If F is a finite projective Λ-module, then $R\Gamma(X_U^{BS}, \mathscr{F}_\rho)$ is represented by a perfect complex of Λ-modules, i.e. a bounded complex of finite projective Λ-modules.*

Proof. X_U^{BS} has a finite triangulation $\mathscr{T}$ consisting of maps $\sigma_\alpha : \Delta^{n_\alpha} \hookrightarrow X_U^{BS}$ where Δ^{n_α} is the simplex of dimension n_α, satisfying the usual properties (the intersection of the images of two such maps is another one, etc.). If $\sigma_\alpha \in \mathscr{T}$ then define

$$\operatorname{star}(\sigma_\alpha) := \bigcup_{\operatorname{im}(\sigma_\beta)\supset\operatorname{im}(\sigma_\alpha)} \operatorname{im}(\sigma_\beta).$$

For example, if your triangulation is a vertex surrounded by four adjacent triangles, the star of one of those triangles is the triangle itself, while the star of the vertex is all of those four triangles. The star construction reverses inclusions: if $\operatorname{im}(\sigma_\alpha) \subset \operatorname{im}(\sigma_\beta)$ then $\operatorname{star}(\sigma_\alpha) \supset \operatorname{star}(\sigma_\beta)$. Also, $\operatorname{star}(\sigma_\alpha)$ is closed and contractible, since each point in the star can be connected to a point in σ_α and then you can scale it down to the point.

Let $\mathscr{G}_i = \bigoplus_{n_\alpha=i} \mathscr{F}|_{\operatorname{star}(\sigma_\alpha)}$. This is acyclic for $R\Gamma$ because $\operatorname{star}(\sigma_\alpha)$ is contractible, so the global sections are

$$R\Gamma(\mathscr{G}_i) = \Gamma(\mathscr{G}_i) = \bigoplus_{n_\alpha=i} \Gamma(\operatorname{star}(\sigma_\alpha), \mathscr{F}|_{\operatorname{star}(\sigma_\alpha)}) = \bigoplus_{n_\alpha=i} F.$$

We have a map $\mathscr{F}_\rho \to \mathscr{G}_0$, because for any closed subset Y, we have a restriction map $\mathscr{F} \to \mathscr{F}|_Y$ obtained from the natural map $\mathscr{F}(W) \to \varinjlim_{W'\supset Y\cap W} \mathscr{F}(W')$ (coming from the fact that a sufficiently small W' is contained in W). In fact, we have maps

$$0 \to \mathscr{F}_\rho \to \mathscr{G}_0 \to \mathscr{G}_1 \to \mathscr{G}_2 \to \cdots$$

where $\mathscr{G}_i \to \mathscr{G}_{i+1}$ is induced from the maps $\mathscr{F}|_{\operatorname{star}(\sigma_\alpha)} \to \mathscr{F}|_{\operatorname{star}(\sigma_\beta)}$ which are 0 if $\sigma_\beta \not\supset \sigma_\alpha$ and $\pm$ the restriction map if $\sigma_\beta \supset \sigma_\alpha$, with the $\pm$ determined by $(-1)^j$ where α omits vertex j of β (assuming we numbered the vertices at the beginning).

We claim that this sequence is exact. We can check this on stalks. Any x lies in the interior of a unique simplex σ_γ. Suppose $x \in \sigma_\gamma(\Delta^{n_\gamma} \setminus \partial\Delta^{n_\gamma})$. Then $x \in \operatorname{star}(\sigma_\alpha)$ if and only if $\sigma_\gamma \subset \operatorname{star}(\sigma_\alpha)$, which is true if and only if $\sigma_\alpha \subset \operatorname{star}(\sigma_\gamma)$ (i.e. both of these happen precisely when σ_α and σ_γ are contained in a common simplex), so the stalks are

$$0 \to \mathscr{F}_{\rho,x} \to \bigoplus_{\substack{n_\alpha=0\\ \sigma_\alpha\subset\operatorname{star}(\sigma_\gamma)}} \mathscr{F}_{\rho,x} \to \bigoplus_{\substack{n_\alpha=1\\ \sigma_\alpha\subset\operatorname{star}(\sigma_\gamma)}} \mathscr{F}_{\rho,x} \to \cdots$$

where each $\mathscr{F}_{\rho,x}$ is F. This is the same as

$$F \otimes (0 \to \mathbb{Z} \to (\text{the complex computing } H^\bullet(\operatorname{star}(\sigma_\gamma), \mathbb{Z})),$$

which is exact because $\operatorname{star}(\sigma_\gamma)$ is contractible so its cohomology vanishes except in degree 0 where it is $\mathbb{Z}$, and all terms are free over $\mathbb{Z}$ so tensoring with F preserves exactness. We conclude that $R\Gamma(X_U^{BS}, \mathscr{F}_\rho)$ is computed by

$$\Gamma(X_U^{BS}, \mathscr{G}_0) \to \Gamma(X_U^{BS}, \mathscr{G}_1) \to \cdots$$

or (pulling out direct sums)

$$\bigoplus_{n_\alpha=0} \Gamma(\operatorname{star}(\sigma_\alpha), \mathscr{F}_\rho) \to \bigoplus_{n_\alpha=1} \Gamma(\operatorname{star}(\sigma_\alpha), \mathscr{F}_\rho) \to \cdots$$

or (since the stars are contractible and $\mathscr{F}_\rho$ is locally constant)

$$\bigoplus_{n_\alpha=0} F \to \bigoplus_{n_\alpha=1} F \to \cdots$$

where the maps are the usual ones for simplicial complexes. This is the finitely generated/projective complex we were looking for. $\square$

Last time, we noted that when $G(\mathbb{R})$ is compact mod center, this theory becomes that of the cohomology of locally constant sheaves on a finite set of points, where there's only an H^0 which is the space of automorphic forms we've discussed in past classes.

5.2 Hecke/adelic group action

Let $\Delta \subset G(\mathbb{A}^\infty)$ be an open sub-semigroup containing 1, and suppose $\rho : \Delta \times F \to F$ is a continuous representation (F is still totally disconnected). Suppose $g \in \Delta$ and $U, V \subset \Delta$ are open subgroups with $U \subset gVg^{-1}$. We have $g : X_U^{BS} \to X_V^{BS}$, and $\mathscr{F}_\rho$ on X_V^{BS} can be pulled back to $g^*\mathscr{F}_\rho$ on X_U^{BS}. We claim that there is a natural map $g^*\mathscr{F}_\rho \to \mathscr{F}_\rho$ over X_U^{BS}. We have

$$(g^*\mathscr{F}_\rho)(W) = \mathscr{F}_\rho(Wg) = \{f : (\text{preimage of } Wg) \to F \cdots\}$$

where the preimage is under the covering $G(\mathbb{Q})\backslash(G(\mathbb{A}^\infty) \times X^{BS}) \to X_V^{BS}$, and the claimed natural map takes f to

$$(x \mapsto \rho(g)f(xg)) \in \mathscr{F}_\rho(W).$$

A map in one direction on the spaces with a backwards map on the sheaves gives a backwards map on cohomology, so we get

$$g : R\Gamma(X_V^{BS}, \mathscr{F}_\rho) \to R\Gamma(X_U^{BS}, \mathscr{F}_\rho).$$

We have $g_1 \circ g_2 = g_1 g_2$ when this makes sense, so it is a left action (the action on spaces is a right action, cohomology reverses maps on spaces, and the action on sheaves is a left action, so we get a left action overall). If $u \in U$, u acts trivially on $R\Gamma(X_U^{BS}, \mathscr{F}_\rho)$.

$R\Gamma(X_V^{BS}, \mathscr{F}_\rho)$ lies in $D^b(Ab)$ or $D^b(\Lambda - Mod)$, and we will actually put it something with slightly more structure. If $V \subset U \subset \Delta$ are open compacts, let $\pi_{V/U} : X_V^{BS} \to X_U^{BS}$ be the forgetful map (this is the same as the map $1 : X_U^{BS} \to X_V^{BS}$ defined earlier). Let

$$I_{V\backslash U} = \{\varphi : V\backslash U \to \mathbb{Z}\}$$

with an action of U given by $(u\varphi)(u') = \varphi(u'u)$. If V is normal in U, then $V\backslash U$ also acts on $I_{V\backslash U}$ by $(u.\varphi)(u') = \varphi(u^{-1}u')$, and this commutes with the previous action.

Lemma 5.2.1.

$$R^i\pi_{V/U,*}\mathscr{F}_\rho = \begin{cases} (0) & i > 0 \\ \mathscr{F}_{I_{V\backslash U} \otimes_{\mathbb{Z}} \rho} & i = 0 \end{cases}$$

where the (0) *for* $i > 0$ *is just because* $\pi_{V/U}$ *is finite. Hence*

$$R\Gamma(X_U^{BS}, \mathscr{F}_{I_{V\backslash U} \otimes \rho}) \cong R\Gamma(X_V^{BS}, \mathscr{F}_\rho).$$

Furthermore, if V *is normal in* U*, then* $V\backslash U$ *acts on* $R\Gamma(X_V^{BS}, \mathscr{F}_\rho)$ *and this action corresponds to the . action on* $R\Gamma(X_U^{BS}, \mathscr{F}_{I_{V\backslash U} \otimes \rho})$.

Proof. We will write π for both $G(\mathbb{Q})\backslash(G(\mathbb{A}^\infty)\times X^{BS}) \to X_V^{BS}$ and $G(\mathbb{Q})\backslash(G(\mathbb{A}^\infty)\times X^{BS}) \to X_U^{BS}$. Then

$$\begin{aligned}(\pi_{V/U,*}\mathscr{F}_\rho)(W) &= \mathscr{F}_\rho(\pi_{V/U}^{-1}W)\\ &= \{f : \pi^{-1}W \to F \mid f \text{ continuous}, f(xv) = v^{-1}f(x)\forall x \in \pi^{-1}W, v \in V\}\\ &\cong \{f' : \pi^{-1}W \to Map(V\backslash U, F) \mid f' \text{ continuous}, f'(xu) = u^{-1}f'(x)\}\end{aligned}$$

where the U-action on $Map(V\backslash U, F)$ is $(u\varphi)(u') = u\varphi(u'u)$, and last two things correspond via

$$\begin{aligned}f'(x)(u) &= u^{-1}f(xu^{-1})\\ f(x) &= f'(x)(1).\end{aligned}$$

Now we have an isomorphism

$$\begin{aligned}I_{V\backslash U} \otimes F &\xrightarrow{\sim} Map(V\backslash U, F)\\ \varphi \otimes a &\mapsto (u \mapsto \varphi(u)a)\end{aligned}$$

compatibly with the U-actions. Consequently

$$(\pi_{V/U,*}\mathscr{F}_\rho)(W) \cong \mathscr{F}_{I_{V\backslash U}\otimes\rho}(W).$$

□

Let

$$\begin{aligned}\mathrm{tr} : I_{V\backslash U} &\to \mathbb{Z}\\ \varphi &\mapsto \sum_{u\in V\backslash U} \varphi(u).\end{aligned}$$

This is U-equivariant for the trivial action on $\mathbb{Z}$. Then we get a map

$$R\Gamma(X_V^{BS}, \mathscr{F}_\rho) \cong R\Gamma(X_U^{BS}, \mathscr{F}_{I_{V\backslash U}\otimes\rho}) \xrightarrow{\mathrm{tr}\otimes 1} R\Gamma(X_U^{BS}, \mathscr{F}_\rho)$$

which we'll also call tr (the trace map). Then for open compact subgroups $U, V \subset \Delta$ and $g \in \Delta$, we will define a map

$$[UgV] : R\Gamma(X_V^{BS}, \mathscr{F}_\rho) \xrightarrow{g} R\Gamma(X_{U\cap gVg^{-1}}^{BS}, \mathscr{F}_\rho) \xrightarrow{tr} R\Gamma(X_U^{BS}, \mathscr{F}_\rho).$$

This depends only on the double coset UgV. To compose $[WhU]$ and $[UgV]$, if we write $WhU = \coprod h_\alpha U$, $UgV = \coprod g_\beta V$, then we have

$$\coprod_{\alpha,\beta} h_\alpha g_\beta V = \coprod_i Wk_iV$$

for some k_i, so that

$$[WhU][UgV] = \sum_i [Wk_iU].$$

5.3 Extra structure on complexes

Now let's go back to the situation where V is normal in U (both open compact subgroups). Let F be a Λ-module. We saw that

$$R\Gamma(X_U^{BS}, \mathscr{F}_{I_{V\backslash U}\otimes\rho}) \cong R\Gamma(X_V^{BS}, \mathscr{F}_\rho) \in D^b(\Lambda - Mod).$$

But $I_{V\backslash U} \otimes \rho$ is a $\Lambda[V\backslash U]$-module, so

$$R\Gamma(X_U^{BS}, \mathscr{F}_{I_{V\backslash U}\otimes\rho}) \in D^b(\Lambda[V\backslash U] - Mod).$$

We have a forgetful map $D^b(\Lambda[V\backslash U] - Mod) \to D^b(\Lambda - Mod)$. But we can actually then think of $R\Gamma(X_V^{BS}, \mathscr{F}_\rho)$ as an element of $D^b(\Lambda[V\backslash U] - Mod)$; in this case we will write it as $R\Gamma(X_V^{BS}, \mathscr{F}_\rho)_U$.

Lemma 5.3.1. *Suppose F is finite projective over Λ.*

1. *$R\Gamma(X_V^{BS}, \mathscr{F}_\rho)_U$ is represented by a perfect complex of $\Lambda[V\backslash U]$-modules (same proof).*
2. *$R\Gamma(X_U^{BS}, \mathscr{F}_\rho) \cong RHom_{\Lambda[V\backslash U]}(\Lambda, R\Gamma(X_V^{BS}, \mathscr{F}_\rho)_U)$.*

In particular, if Λ is a $\mathbb{Q}$-algebra, then $H^i(X_U^{BS}, \mathscr{F}_\rho) = H^i(X_V^{BS}, \mathscr{F}_\rho)^U$ (since U/V, being a finite group, has no cohomology). In general, you will have a spectral sequence involving the higher U/V-cohomology.

Proof. Sketch of Part 2: in degree 0, we can take global sections and fixed points in any order, so we have

$$R\Gamma(X_U^{BS}, \underline{RHom}_{\Lambda[V\backslash U]}(\underline{\Lambda}, R\pi_{V/U,*}\mathscr{F}_\rho)) \cong R\mathrm{Hom}_{\Lambda[V\backslash U]}(\Lambda, R\Gamma(X_U^{BS}, R\pi_{V/U,*}\mathscr{F}_\rho))$$

where we know that

$$R\Gamma(X_U^{BS}, R\pi_{V/U,*}\mathscr{F}_\rho) = R\Gamma(X_V^{BS}, \mathscr{F}_\rho)_U.$$

So we need to show that

$$\mathscr{F}_\rho \xrightarrow{\sim} \underline{RHom}_{\Lambda[V\backslash U]}(\underline{\Lambda}, R\pi_{V/U,*}\mathscr{F}_\rho).$$

We know that $R\pi_{V/U,*}\mathscr{F}_\rho = \mathscr{F}_{I_{V\backslash U}\otimes\rho}$, so it suffices to check on the level of modules that

$$F \xrightarrow{\sim} R\mathrm{Hom}_{\Lambda[V\backslash U]}(\Lambda, I_{V\backslash U} \otimes F).$$

Since F is finite projective over Λ, it is a direct summand of a finite free Λ-module, and using the compatibility of the desired isomorphisms with direct sums, we can reduce to the case $F = \Lambda$. Then we want to check that

$$\Lambda \xrightarrow{\sim} R\mathrm{Hom}_{\Lambda[V\backslash U]}(\Lambda, I_{V\backslash U}).$$

In fact we claim that

$$R\mathrm{Hom}_{\Lambda[V\backslash U]}(\Lambda, I_{V\backslash U}) = R\mathrm{Hom}_\Lambda(\Lambda, \Lambda) = \Lambda$$

and more generally that

$$R\mathrm{Hom}_\Lambda(M,\Lambda) \cong R\mathrm{Hom}_{\Lambda[V\backslash U]}(M, I_{V\backslash U})$$

for $M \in D^b(\Lambda[V\backslash U]-Mod)$. This would follow from

$$\mathrm{Hom}_\Lambda(M,N) \cong \mathrm{Hom}_{\Lambda[V\backslash U]}(M, I_{V\backslash U})$$

which are isomorphic via $\varphi \mapsto \tilde{\varphi}$ where $\varphi(m) = \tilde{\varphi}(m)(1)$ and $\tilde{\varphi}(m)(u) = \varphi(u^{-1}m)$. $\square$

The point is that if we know what happens for small subgroups then we know what happens for all subgroups.

For example, suppose $V \subset \Delta$ is a closed subgroup (e.g. $V = \{1\}$). We define

$$H^i(X_V^{BS}, \mathscr{F}_\rho) = \varinjlim_{\substack{\Delta \supset U \supset V \\ U \text{ open}}} H^i(X_U^{BS}, \mathscr{F}_\rho)$$

which has an action of the normalizer $Norm_\Delta(V)$ of V in Δ (because the normalizer permutes the possible Us). If Λ is a $\mathbb{Q}$-algebra and V is normal in U, then this gives

$$H^i(X_U^{BS}, \mathscr{F}_\rho) = H^i(X_V^{BS}, \mathscr{F}_\rho)^U$$

where the RHS again has an action of $Norm_\Delta(V)$ (for example if $V = \{1\}$ we get an action of all of Δ). Then for $g \in Norm_\Delta(V)$, we have

$$\begin{array}{ccc}
H^i(X_U^{BS}, \mathscr{F}_\rho) & \xrightarrow{[U'gU]} & H^i(X_{U'}^{BS}, \mathscr{F}_\rho) \\
\Big\downarrow = & & \Big\downarrow = \\
H^i(X_V^{BS}, \mathscr{F}_\rho)^U & \xrightarrow{\sum g_i} & H^i(X_V^{BS}, \mathscr{F}_\rho)^{U'}
\end{array}$$

where $U'gU = \coprod g_i U$. This gives the Hecke operator action on the fixed points of the cohomology for given open subgroups.

5.4 Starting automorphic forms

These sheaves and the Hecke actions on them are the analogue of spaces of l-adic automorphic forms. For zero-dimensional locally symmetric spaces, we used some changes of variables to relate these to classical automorphic forms; now we want to do the same thing for these sheaves. We will now have to restrict the kinds of representations we consider.

Lemma 5.4.1. *Suppose $L/\mathbb{Q}_l$ is algebraic and $F = F_1 \otimes_L F_2$ where each F_i is a finite dimensional L-vector space, and $\rho = \rho_1 \otimes \rho_2$ where ρ_1 is an algebraic representation of G over L, thought of as a representation of U via $\rho_1(u) = \rho_1(u_l)$, and ρ_2 is a representation of U with open kernel (smooth).*

Let $\sigma : G(\mathbb{A}^\infty)/\ker \rho_2 \times X^{BS} \to X_U^{BS}$. Then we have

$$\mathscr{F}_\rho(W) = \{\tilde{f} : \sigma^{-1}W \to F \mid \tilde{f} \text{ locally constant}, \tilde{f}(\gamma x u) = \rho_1(\gamma) \otimes \rho_2(u)^{-1} \tilde{f}(x)\}.$$

(Recall that in the zero-dimensional locally symmetric space case, the process for going from l-adic automorphic forms to archimedean automorphic forms went through an intermediate stage where you turn the $G(\mathbb{Q})$-invariance into a different natural action of $G(\mathbb{Q})$. This is the same idea.)

Proof. If $\pi : G(\mathbb{Q})\backslash(G(\mathbb{A}^\infty) \times X^{BS}) \to F$, our original expression for $\mathscr{F}_\rho(W)$ was

$$\mathscr{F}_\rho(W) = \{f : \pi^{-1}W \to F \mid f \text{ continuous}, f(xu) = \rho(u)^{-1}f(x)\}$$

and the bijection between f and $\tilde{f}$ is $\tilde{f}(x) = (\rho_1(x_l) \otimes 1)f(x)$. □

Next time, we will do the analogous second part where we take $\rho_1(x_\infty)$ back out, thereby getting automorphic forms.

In the case $GL_2(\mathbb{Q})$, what we have written here is the Betti cohomology of the open modular curve with coefficients in an l-adic sheaf. Probably $X_U \hookrightarrow X_U^{BS}$ is a homotopy equivalence, and in any case $H^\bullet(X_U, \mathscr{F}_\rho) = H^\bullet(X_U^{BS}, \mathscr{F}_\rho)$, where $\mathscr{F}_\rho$ is typically an l-adic sheaf. You could consider bigger coefficients—instead of having them be finite rank over $\mathbb{Q}_l$, they could be e.g. a completed group algebra, and then you'd recover Emerton's completed cohomology. But we won't need that.

6 April 15: classical automorphic forms.

6.1 Translation from l-adic forms

Recall that we had a connected linear algebraic group $G/\mathbb{Q}$, an open compact subgroup $U \subset G(\mathbb{A}^\infty)$, and a continuous representation $\rho : U \times F \to F$ where F is a totally disconnected topological abelian group (or Λ-module for some ring Λ). We defined a sheaf $\mathscr{F}_\rho$ on X_U^{BS} in terms of the uniformization map (a U-torsor) $\pi : G(\mathbb{Q})\backslash(G(\mathbb{A}^\infty) \times X^{BS}) \to X_U^{BS}$ by

$$\mathscr{F}_\rho(W) = \{f : \pi^{-1}W \to F \mid f \text{ continuous}, f(xu) = u^{-1}f(x) \forall x \in W, u \in U\}.$$

We defined actions on this of $g \in G(\mathbb{A}^\infty)$, $[UgV]$, etc. We showed that $\mathscr{F}_\rho$ is locally constant. We showed that its cohomology is the same whether or not you take BS, and has nice finiteness properties because BS has a finite triangulation.

Then we started to set up an alternative definition in order to compare l-adic forms with archimedean forms. We saw that if $L/\mathbb{Q}_l$ is an algebraic extension, $F = F_1 \otimes_L F_2$ is a tensor product of finite dimensional L-vector spaces, and $\rho = \rho_1 \otimes \rho_2$ where ρ_1 is an algebraic representation of G over L (hence a representation of $G(\mathbb{Q}_l)$ over L) and ρ_2 has open kernel, then we can give a second description of $\mathscr{F}_\rho$ as follows. We used the map $\sigma : G(\mathbb{A}^\infty)/\ker \rho_2 \times X^{BS} \to X_U^{BS}$ (which is already interesting if ρ_2 is trivial and $\ker \rho_2 = U$, so keep that example in mind) and defined

$$\begin{aligned}\mathscr{F}_\rho(W) = \{\tilde{f} : \sigma^{-1}W \to F \mid &\tilde{f} \text{ locally constant},\\ &\tilde{f}(\gamma x u) = \rho_1(\gamma) \otimes \rho_2(u)^{-1}\tilde{f}(x) \forall \gamma \in G(\mathbb{Q}), x \in \sigma^{-1}W, u \in U\}\end{aligned}$$

which is in bijection with the original definition via $\tilde{f}(x) = (\rho_1(x_l) \otimes 1)f(x)$.

Now let's switch to the ∞ component. This is not entirely canonical. Fix $x_0 \to X$, and for $V \subset U$ open, let

$$\begin{aligned}\tau : G(\mathbb{Q})\backslash(G(\mathbb{A}^\infty)/V \times G(\mathbb{R})) &\to X_U \\ (g,h) &\mapsto [(g, hx_0)].\end{aligned}$$

(Note that we can't do this on the Borel-Serre boundary anymore.) The fibers are $U/V \times \tilde{U}_{x_0}$, where we recall that $\tilde{U}_{x_0}$ is the (essentially) maximal compact of $G(\mathbb{R})$ stabilizing x_0. So in the first definition we were modding out by some open subgroup of $G(\mathbb{Q}_l)$, then we modded out by $G(\mathbb{Q})$, and now we're modding out by a maximal compact of $G(\mathbb{R})$.

If ρ is a smooth representation of $U/V \times \tilde{U}_{x_0}$ on a finite dimensional $\mathbb{C}$-vector space H, we define a smooth vector bundle $\mathscr{H}_\rho/X_U$ by

$$\mathscr{H}_\rho(W) = \{\varphi : \tau^{-1}W \to H \mid \varphi \text{ smooth}, \varphi(gu) = \rho(u)^{-1}\varphi(g) \forall u \in U \times \tilde{U}_{x_0}, g \in \tau^{-1}W\}.$$

If ρ is in fact a representation of $U/V \times G(\mathbb{R})$, then we can define connections and take locally constant sections, giving the locally constant sheaf

$$\mathscr{H}_\rho^\nabla(W) = \{\varphi \in \mathscr{H}_\rho(W) \mid X\varphi = 0 \forall X \in \operatorname{Lie} G(\mathbb{R})/\operatorname{Lie} \tilde{U}_{x_0}\}$$

where

$$(X\varphi)(g) = \frac{d}{dt}\left(\rho(e^{tX})\varphi(ge^{tX})\right)|_{t=0}.$$

Lemma 6.1.1. *Suppose $F = F_1 \otimes_L F_2$ where the F_i are finite-dimensional L-vector spaces and $\rho = \rho_1 \otimes \rho_2$ where ρ_1 is an algebraic representation of G over L and ρ_2 is a smooth representation of U. Fix $\iota : \overline{L} \xrightarrow{\sim} \mathbb{C}$ (as we have to do now that we're working with essentially archimedean constructions; recall that algebraically closed fields of the same transcendence degree are isomorphic). Define*

$$\rho(\infty) = \rho_1 \otimes \rho_2 \to F \otimes_{L,\iota} \mathbb{C}$$

where we think of ρ_1 as a representation of G over $\mathbb{C}$ via ι, hence a representation of $G(\mathbb{R})$. Then

$$\mathscr{F}_\rho \otimes_{L,\iota} \mathbb{C} \cong \mathscr{H}^\nabla_{\rho(\infty)}.$$

Proof. We will go between f and $\tilde{f}$ as before, and between $\tilde{f}$ and φ via

$$\varphi(g) = (\rho_1(g_\infty)^{-1} \otimes 1)\tilde{f}(gx_0).$$

(By gx_0 we mean $(g^\infty, g_\infty x_0)$.) The $X\varphi = 0$ condition comes from the fact that $\tilde{f}$ is locally constant. That is, we have

$$\begin{aligned}(X\varphi)(g) &= \frac{d}{dt}((\rho_1(e^{tX}) \otimes 1)(\rho_1(e^{-tX}g_\infty^{-1}) \otimes 1)\tilde{f}(g^\infty, g_\infty e^{tX}x_0))|_{t=0} \\ &= \frac{d}{dt}((\rho_1(g_\infty)^{-1} \otimes 1)\tilde{f}(g^\infty, g_\infty e^{tX}x_0))|_{t=0} \\ &= \frac{d}{dt}(\varphi(g))|_{t=0} = 0\end{aligned}$$

because $\tilde{f}$ is locally constant and so $\tilde{f}(g^\infty, g_\infty e^{tX}x_0)$ is eventually the same as $\tilde{f}(g^\infty, g_\infty x_0)$ for small t. □

6.2 Franke's theorem

Now that we are in the setting of vector bundles with connections, we might hope to compute cohomologies using de Rham cohomology. But this is hard to do right now because while $\mathscr{F}_\rho$ was defined over X including the Borel-Serre boundary, $\mathscr{H}^\nabla_{\rho(\infty)}$ has so far only been defined over the open locally symmetric space. This issue was resolved by Franke.

Theorem 6.2.1 (Franke)**.** *Keep the assumptions in the lemma and suppose that G is reductive. Then we have the following formula for the de Rham cohomology of X_U in terms of the $(\mathfrak{g}, K)$-cohomology of automorphic forms (to be recalled later):*

$$H^i(X_U, \mathscr{F}_\rho) \otimes_{L,\iota} \mathbb{C}$$

$$\cong \mathrm{Hom}_U\left(\rho_2^\vee, H^i\left((\mathrm{Lie}\, M_G) \otimes_{\mathbb{R}} \mathbb{C}, U_{x_0}, (\mathscr{A}(G(\mathbb{Q})\backslash G(\mathbb{A})) \otimes \rho_1)^{A_G(\mathbb{R})^+}\right)\right)$$

where we recall that $M_G = \bigcap_{\chi: G \to \mathbb{G}_m/\mathbb{Q}} \ker \chi^2$.

(Richard wishes the $A_G(\mathbb{R})^+$ could be moved to the earlier terms $\mathrm{Lie}\, M_G$ and U_{x_0}, but it doesn't seem to work.)

Okay, let's recall $(\mathfrak{g}, K)$-cohomology. Let M be a $(\mathfrak{g}, K)$-module, where G is a real Lie group, $\mathfrak{g} = (\mathrm{Lie}\, G) \otimes_{\mathbb{R}} \mathbb{C}$, and $K \subset G$ is a compact subgroup. So M is a vector space with an action of $\mathfrak{g}$ and a locally finite continuous/smooth action of K. That means that for $m \in M$, $\langle Km \rangle_{\mathbb{C}}$ is finite-dimensional with continuous K-action. The two actions are compatible, meaning that the differential of the K-action is the action of $\mathrm{Lie}\, K \subset \mathfrak{g}$, and $\mathrm{ad}(k)(X)$ acts as $k \circ X \circ k^{-1}$.

Then $H^\bullet(\mathfrak{g}, K, M)$ comes from the right derived functor of the part of M where G acts trivially, or equivalently the cohomology of the complex

$$\mathrm{Hom}_K(\wedge^\bullet \mathfrak{g}/(\mathrm{Lie}\, K)_{\mathbb{C}}, M)$$

where

$$\begin{aligned}(d\varphi)(X_1 \wedge \cdots \wedge X_{n+1}) = &\sum_{i=1}^{n+1} (-1)^{i+1} X_i \varphi(X_1 \wedge \cdots \widehat{X}_i \cdots \wedge X_{n+1}) \\ &+ \sum_{1 \le i < j \le n+1} (-1)^{i+j} \varphi([X_i, X_j] \wedge X_1 \cdots \widehat{X}_i \cdots \widehat{X}_j \cdots \wedge X_{n+1})\end{aligned}$$

where by $\widehat{X}_i$ we mean to omit that term.

We can simplify this a little by splitting up the space of automorphic forms.

6.3 Decomposing automorphic forms

Definition 6.3.1. Let G be reductive over $\mathbb{Q}$. We call two parabolic subgroups of G associated if they have Levi components which are conjugate by $G(\mathbb{Q})$.

Example 6.3.2. If $G = GL_3$, the parabolics

$$\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \text{ and } \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

are not conjugate, but are associated, because the natural choices of Levi components are both some permutation of $GL_1 \times GL_2$ and can be conjugated to each other by permuting basis elements.

Definition 6.3.3. If P, P' are parabolic subgroups, π is a(n irreducible) cuspidal automorphic representation of $L_P(\mathbb{A})$, and π' is a cuspidal automorphic representation of $L_{P'}(\mathbb{A})$, then we say (P, π) and (P', π') are equivalent if there is $\gamma \in G(\mathbb{Q})$ conjugating a lift $\tilde{L}_P$ of L_P to a lift $\tilde{L}_{P'}$ of $L_{P'}$ such that $\pi \cong \pi' \circ \mathrm{conj}_\gamma$.

By a "cuspidal data" Φ for G we mean an equivalence class of such pairs $[(P, \pi)]$.

(Note that we are asking for just an isomorphism, not equality in the space of automorphic forms, in the event that multiplicity one doesn't hold.)

We have a decomposition (whose details we won't define)

$$\mathscr{A}(G(\mathbb{Q})\backslash G(\mathbb{A})) = \bigoplus_{\Phi \text{ cuspidal data}} \mathscr{A}_\Phi(G(\mathbb{Q})\backslash G(\mathbb{A})).$$

What's important to know is that every irreducible subquotient of $\mathscr{A}_{[P,\pi]}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ is an irreducible subquotient of $n - \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \pi$. (Again, we might have isomorphic representations appearing in different summands. Also the summands are not semisimple.) The cuspidal part is

$$\mathscr{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A})) = \bigoplus_{[(G,\pi)]} \mathscr{A}_\Phi(G(\mathbb{Q})\backslash G(\mathbb{A})).$$

π is irreducible and hence has an infinitesimal character. So as a consequence of being an irreducible subquotient of $n - \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \pi$, every irreducible subquotient of $\mathscr{A}_\Phi(G(\mathbb{Q})\backslash G(\mathbb{A}))$ has the same infinitesimal character $\xi_\Phi : \mathfrak{Z}_G \to \mathbb{C}$, where

$$\mathfrak{Z}_G = Z(\mathfrak{U}(\mathrm{Lie}\, G(\mathbb{R}) \otimes_\mathbb{R} \mathbb{C}))$$

where $\mathfrak{U}$ is the universal enveloping algebra and Z means to take the center. Recall that this is a polynomial ring: if $T \subset G$ is a maximal torus, and W is the Weyl group, then we have the Harish-Chandra isomorphism

$$\gamma_G : (\mathrm{Sym}^\bullet(\mathrm{Lie}\, T(\mathbb{R}) \otimes_\mathbb{R} \mathbb{C}))^W \xrightarrow{\sim} \mathfrak{Z}_G.$$

Corollary 6.3.4. *Assumptions as in the theorem. We have*

$$H^i(X_U, \mathscr{F}_\rho) \otimes_{L,\iota} \mathbb{C} \cong \bigoplus_{\substack{\Phi \\ \xi_\Phi = \xi_{\rho_1^\vee}}} \mathrm{Hom}_U\left(\rho_2^\vee, H^i\left((\mathrm{Lie}\, M_G)_\mathbb{C}, U_{x_0}, (\mathscr{A}_\Phi(G(\mathbb{Q})\backslash G(\mathbb{A})) \otimes \rho_1)^{A_G(\mathbb{R})^+}\right)\right).$$

(Note: if $\Phi = [(P, \pi)]$, then since $W_G \supset W_L$, we have a natural inclusion

$$(\mathrm{Sym}^\bullet(\mathrm{Lie}\, T)_\mathbb{C})^{W_G} \hookrightarrow (\mathrm{Sym}^\bullet(\mathrm{Lie}\, T)_\mathbb{C})^{W_L}$$

which means that

$$\xi_\Phi : (\mathrm{Sym}^\bullet(\mathrm{Lie}\, T)_\mathbb{C})^{W_G} \cong^{\gamma_G} \mathfrak{Z}_G \to \mathbb{C}$$

can be described as the restriction of

$$\xi_\pi : (\mathrm{Sym}^\bullet(\mathrm{Lie}\, T)_\mathbb{C})^{W_L} \cong^{\gamma_L} \mathfrak{Z}_L \to \mathbb{C}.)$$

This decomposition has the advantage that we know what the Hecke action on $\mathscr{A}_\varphi$ looks like, but again we have to remember that $(\mathscr{A}_\Phi(G(\mathbb{Q})\backslash G(\mathbb{A})) \otimes \rho_1)^{A_G(\mathbb{R})^+}$ is not semisimple so the homomorphisms from $\rho_2^\vee$ and the cohomology are likely to be complicated. The term $H^i((\mathrm{Lie}\, M_G)_\mathbb{C}, \dots)$ vanishes if the condition $\xi_\Phi = \xi_{\rho_1^\vee}$ does not hold.

Assuming $L = \overline{\mathbb{Q}}_l$ (so all direct summands are defined), we will write $H^i(X_U, \mathscr{F}_\rho)_\Phi$ for the Φ-summand in the above corollary, so that

$$H^i(X_U, \mathscr{F}_\rho) = \bigoplus_{\substack{\Phi \\ \xi_\Phi = \xi_{\rho_1^\vee}}} H^i(X_U, \mathscr{F}_\rho)_\Phi.$$

If $\Phi = [(G, \pi)]$, the failure of semisimplicity is relatively simple. Everything but $A_G(\mathbb{R})^+$ acts semisimply. $A_G(\mathbb{R})^+$, being in the center, acts by a character on ρ_1, so the fixed points $(\mathscr{A}_\Phi(G(\mathbb{Q})\backslash G(\mathbb{A})) \otimes \rho_1)^{A_G(\mathbb{R})^+}$ pick out the subspace of $\mathscr{A}_\Phi(G(\mathbb{Q})\backslash G(\mathbb{A}))$ on which $A_G(\mathbb{R})^+$ acts by the inverse character. So if π occurs with multiplicity m_π in $\mathscr{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}))$ (by which we mean that $\pi^{\oplus m} \hookrightarrow \mathscr{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}))$ if and only if $m \le m_\pi$—i.e. we're counting copies of π as submodules, not subquotients), then

$$H^i(X_U, \mathscr{F}_\rho)_\Phi \otimes_{\overline{\mathbb{Q}}_l, \iota} \mathbb{C} \cong \mathrm{Hom}_U(\rho_2^\vee, \pi^\infty)^{m_\pi} \otimes H^i\left((\mathrm{Lie}\, M_G)_\mathbb{C}, U_{x_0}, \pi_\infty \otimes \rho_1\right)$$

if $\zeta_\pi = \zeta_{\rho_1^\vee}$, and otherwise the LHS is (0). (We could have written $(\pi_\infty \otimes \rho_1)^{A_G(\mathbb{R})^+}$, but since $\zeta_\pi = \zeta_{\rho_1^\vee}$ and $A_G(\mathbb{R})^+$ acts by a character on both factors, it must act by inverse characters on the two factors and thus preserve the whole thing.)

When Φ corresponds to a proper parabolic, things are much more complicated because $\mathscr{A}_\Phi$ is non-semisimple in a complicated way.

6.4 $G = GL_n$

That's all we wanted to say about the cohomology of locally symmetric spaces in complete generality; now we'll specialize to the case $G = GL_n$, because that's where we know the most about the relationship between Galois representations and automorphic forms.

To avoid introducing too many technical conditions, we will work in the following situation, though some of what we say will be more general, and we expect those technical conditions to go away in the future. Fix, probably for the rest of the course, F_0 an imaginary quadratic field, F^+ a totally real field, and $F = F_0 F^+$ a CM field. (Most of what we say should be true for all CM fields, but some of it has only been proven when it contains an imaginary quadratic field.) Let $G = \operatorname{res}_{\mathbb{Z}}^{\mathcal{O}_F} GL_n$, so $G(\mathbb{Q}) = GL_n(F)$ and $G(\mathbb{A}) = GL_n(\mathbb{A}_F)$. Let X be a locally symmetric space for G.

What is $\dim_{\mathbb{R}} X$? The dimension of $G(\mathbb{R})$ is $n^2[F : \mathbb{Q}]$; we subtract the dimension of the split-over-$\mathbb{Q}$ part of the center, which is 1 because the whole center is $\operatorname{res}_{\mathbb{Q}}^F \mathbb{G}_m$; we also subtract the dimension of the maximal compact U_n, which is $n^2[F^+ : \mathbb{Q}]$. We get

$$\dim_{\mathbb{R}} X = n^2[F : \mathbb{Q}] - 1 - n^2[F^+ : \mathbb{Q}] = n^2[F^+ : \mathbb{Q}] - 1$$

(for example $4 \cdot 1 - 1 = 3$ for just an imaginary quadratic field).

Let l be prime and $L/\mathbb{Q}_l$ a finite extension such that for all $\tau : F \hookrightarrow \overline{L}$, $\tau F \subset L$. Let $\mathcal{O} = \mathcal{O}_L$, λ the maximal ideal in $\mathcal{O}$, $\mathbb{F} = \mathcal{O}/\lambda$ a finite field.

What are algebraic representations of G like? Let

$$\lambda \in \mathbb{Z}^n_+ = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\} \subset X^*(\mathbb{G}_m^n).$$

For A a ring, define the representation of $GL_n(A)$

$$V_\lambda(A) = \{f \in A[GL_n] \mid f(bg) = (w_0\lambda)(b)f(g) \forall g \in GL_n, b \in B_n\}$$

where by $A[GL_n]$ we mean regular functions (polynomials) on the scheme GL_n in the sense of algebraic geometry, $B_n \subset GL_n$ is the upper triangular Borel subgroup, w_0 is the longest element of the Weyl group (so $w_0(\lambda_1, \dots, \lambda_n) = (\lambda_n, \dots, \lambda_1)$), and the value of $\mu \in \mathbb{Z}^n$ on an upper triangular matrix with diagonal elements $(b_1, b_2, \dots, b_n)$ is $b_1^{\mu_1} \cdots b_n^{\mu_n}$. The action of $h \in GL_n(A)$ on $V_\lambda(A)$ is $(hf)(g) = f(gh)$. Then V_λ is an algebraic representation of GL_n over $\mathbb{Z}$, $V_\lambda(A)$ is a finite free A-module, and $V_\lambda(A) = V_\lambda(\mathbb{Z}) \otimes_{\mathbb{Z}} A$.

$V_\lambda(\mathbb{Q})$ is the usual irreducible representation of GL_n with highest weight λ.

(In finite characteristic V_λ is no longer necessarily irreducible.)

7 April 20: Hecke operators for GL_n.

7.1 GL_n over a CM field

Recall our notation: F_0 is an imaginary quadratic field, F^+ is a totally real field, and $F = F_0F^+$ the corresponding CM field. (Some of what we say will work for all CM fields, but we will always work with F of this form.) $G = \operatorname{res}_{\mathbb{Z}}^{\mathcal{O}_F} GL_n$ (last time we wrote $\operatorname{res}_{\mathbb{Q}}^F GL_n$, but we should keep the integral structure), so $G(\mathbb{Q}) = GL_n(F)$, and X is a symmetric space for G, with real dimension $[F^+ : \mathbb{Q}]n^2 - 1$ (for example, hyperbolic 3-space for GL_2 over an imaginary quadratic field).

Assume (as we probably didn't last time) that l is a prime which splits in F_0 (not everything we say will depend on this, but the deepest things will). Let $L/\mathbb{Q}_l$ be finite such that for all $\tau : F \hookrightarrow \overline{L}$, $\tau F \subset L$. Let $\mathcal{O} = \mathcal{O}_L$, λ the maximal ideal of $\mathcal{O}$, $\mathbb{F} = \mathcal{O}/\lambda$ the finite residue field. Let

$$\mathbb{Z}^n_+ = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \cdots \geq \lambda_n\} \subset X^*(\mathbb{G}_m^n).$$

For $\lambda \in \mathbb{Z}^n_+$, we defined an algebraic representation V_λ of GL_n, which is defined over $\mathbb{Z}$. Over a field of characteristic 0, V_λ is the irreducible representation of GL_n of highest weight λ.

(Note: the definition we gave last time is not the unique integral structure for this. It is just a way that works consistently over all rings.)

Next, for $\lambda = (\lambda_\tau) \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(F,L)}$, we can define an algebraic representation V_λ of G over $\mathcal{O}$ by

$$V_\lambda(A) = \bigotimes_{\tau: F \hookrightarrow L} V_{\lambda_\tau}(A)$$

for any algebra A over $\mathcal{O}$, where the tensor product is over A, and G acts via the isomorphism

$$G \times_{\mathbb{Z}} \mathcal{O} = \prod_{\tau: F \hookrightarrow L} GL_n.$$

Then $G(\mathbb{Z}_l)$ acts on $V_\lambda(\mathcal{O})$ and $G(\mathbb{Q}_l)$ acts on $V_\lambda(L)$.

Now we want to talk about the Galois representations associated to eigenclasses in the cohomology of the locally symmetric spaces of G. We'll start with a simple version. Let $U \subset G(\mathbb{A}^\infty) = GL_n(\mathbb{A}_F^\infty)$ be a neat open compact subgroup of the form

$$U = U_S \times \prod_{v \notin S} GL_n(\mathcal{O}_{F,v})$$

where S is a finite set of finite places of F; assume that $S^c = S$ (that is, if a place is in S then so is its complex conjugate). Also assume that if $v \in S$ is such that $v|p$ where p is not split in F_0, then S contains all primes above p. Finally assume that the projection of U to $G(\mathbb{Q}_l)$ is contained in $G(\mathbb{Z}_l)$ (this is automatically true if S doesn't contain any primes above l). (We will keep these assumptions for a long time.)

Define the abstract Hecke algebra

$$\begin{aligned} \mathscr{H}^S &= \mathbb{Z}\left[GL_n(\widehat{\mathcal{O}}_F^S)\backslash GL_n\left(\mathbb{A}_F^{S\cup\{\infty\}}\right)/GL_n(\widehat{\mathcal{O}}_F^S)\right] \\ &\cong {\bigotimes_{v\notin S}}' \mathbb{Z}[GL_n(\mathcal{O}_{F,v})\backslash GL_n(F_v)/GL_n(\mathcal{O}_{F,v})] \end{aligned}$$

where the restricted tensor product is with respect to the trivial double coset $[GL_n(\mathcal{O}_{F,v})]$. For $v \notin S$ and $i = 0, \dots, n$, define

$$T_{v,i} = [GL_n(\mathcal{O}_{F,v}) \operatorname{diag}(\pi_v, \dots, \pi_v, 1, \dots, 1) GL_n(\mathcal{O}_{F,v})] q_v^{i(i-1)/2}$$

where there are i copies of π_v and $(n-i)$ copies of 1. Here π_v is a uniformizer at v, meaning that π_v generates the maximal ideal in $\mathcal{O}_{F,v}$, and $q_v = \#\mathcal{O}_{F,v}/(\pi_v) = \#k(v)$. Let

$$P_v(X) = \sum_{i=0}^n (-1)^i T_{v,i} X^{n-i} \in \mathscr{H}^S[X].$$

(where we think of $T_{v,i}$ inside $\mathscr{H}^S$ via the above tensor product decomposition of $\mathscr{H}^S$).

Lemma 7.1.1. *If π is an irreducible smooth representation of $GL_n(F_v)$ over $\mathbb{C}$, then*

$$\pi^{GL_n(\mathcal{O}_{F,v})} \neq (0) \textit{ if and only if } \operatorname{rec}\left(\pi |\det|_v^{(1-n)/2}\right) \textit{ is unramified.}$$

In this case, $\pi^{GL_n(\mathcal{O}_{F,v})}$ is 1-dimensional, so that $\mathbb{Z}[GL_n(\mathcal{O}_{F,v})\backslash GL_n(F_v)/GL_n(\mathcal{O}_{F,v})]$ acts on it by a character θ_π, and $\theta_\pi(P_v(X)) \in \mathbb{C}[X]$ is the characteristic polynomial of

$$\mathrm{rec}\left(\pi|\det|_v^{(1-n)/2}\right)(\mathrm{Frob}_v).$$

(Recall that an unramified representation is determined up to Frobenius semisimplification by this characteristic polynomial.)

The main thing about this lemma is to pay attention to the normalizations. Note that when we say a WD rep is unramified we mean that $N = 0$ in addition to the representation of the Weil group being unramified. If you allow N to be nonzero in the WD rep then the correspondence would be with Iwahori-fixed vectors instead.

7.2 Structure of the derived Hecke algebra

Given $\lambda \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(F,L)}$, we get a representation V_λ of $G(\mathbb{Z}_l)$ over $\mathcal{O}$, therefore a locally constant sheaf $\mathscr{F}_\lambda$ of $\mathcal{O}$-modules over X_U^{BS}. Look at $\Gamma(X_U^{BS}, \mathscr{F}_\lambda) \in Ob(D^b(\mathcal{O}))$. (Recall that $H^i(X_U^{BS}, \mathscr{F}_\lambda)$ is a finitely generated $\mathcal{O}_F$-module for all i, which is (0) if $i \neq [0, [F^+ : \mathbb{Q}]n^2-1]$.) We described a map

$$\mathscr{H}^S \to \mathrm{End}_{D^b(\mathcal{O})}(\Gamma(X_U^{BS}, \mathscr{F}_\lambda)).$$

We will denote the $\mathcal{O}$-algebra generated by the image by $\mathbb{T}^S(U, \lambda)$. This is commutative because $\mathscr{H}^S$ is. Two less obvious facts:

- $\mathbb{T}^S(U, \lambda)$ is finitely generated as an $\mathcal{O}$-module.
- Consider the natural map

  $$\mathbb{T}^S(U, \lambda) \to \mathrm{End}_{\mathcal{O}}\left(\bigoplus_i H^i(X_U^{BS}, \mathscr{F}_\lambda)\right)$$

  given by applying the H^i functor to $\Gamma(X_U^{BS}, \mathscr{F}_\lambda)$, taking the induced "naive" action of $\mathbb{T}^S(U, \lambda)$ on each individual cohomology group $H^i(X_U^{BS}, \mathscr{F}_\lambda)$, and adding them up. Then we have

  $$\ker\left(\mathbb{T}^S(U, \lambda) \to \mathrm{End}_{\mathcal{O}}\left(\bigoplus_i H^i(X_U^{BS}, \mathscr{F}_\lambda)\right)\right)^{N-1} = (0)$$

  where $N = [F^+ : \mathbb{Q}]n^2$.

Both of these follow from the following general lemma.

Lemma 7.2.1. *Let A be a noetherian ring and let $C, D \in D^b(A)$.*

1. *If $H^\bullet(C)$ and $H^\bullet(D)$ are finitely generated A-modules, then $\mathrm{Hom}_{D^b(A)}(C, D)$ is a finitely generated A-module.*
2. $\ker(\mathrm{End}_{D^b(A)}(C) \to \mathrm{End}_A(\bigoplus_i H^i(C)))^{N-1} = (0)$ *where $N = \#\{i \mid H^i(C) \neq (0)\}$.*

In particular, not only $\mathbb{T}^S(U,\lambda)$ but even $\mathrm{End}_{D^b(\mathcal{O})}(\Gamma(X_U^{BS},\mathscr{F}_\lambda))$ is finitely generated as an $\mathcal{O}$-module, since we previously saw that the cohomology of $\Gamma(X_U^{BS},\mathscr{F}_\lambda)$ is finitely generated (since X_U^{BS} had a finite triangulation and $\mathscr{F}_\lambda$ was a sheaf of finitely generated $\mathcal{O}$-modules).

It turns out that pretty much everything we do will be modulo an unidentified nilpotent ideal whose degree of nilpotency is bounded in terms of $[F:\mathbb{Q}]$ and n, so we wouldn't have lost anything by defining $\mathbb{T}^S(U,\lambda)$ inside $\mathrm{End}_{\mathcal{O}}(\bigoplus_i H^i(X_U^{BS},\mathscr{F}_\lambda))$, but our definition is probably a bit more natural.

Proof. 1. Induction on $N=\#\{i \mid H^i(C)\neq(0)\}$ and $M=\#\{i \mid H^i(D)\neq(0)\}$. Let i be maximal such that $H^i(C)\neq(0)$. We have $\tau^{\geq i}C\cong H^i(C)[-i]$, that is a quasi-isomorphism $H^i(C)[-i]\to\tau^{\geq i}C$; to be concrete, if C is represented by C^i, $\tau^{\geq i}C^i$ is represented by

$$\cdots 0\to 0\to C^i/\operatorname{im}C^{i-1}\to C^{i+1}\to C^{i+2}\to\cdots$$

and $H^i(C)[-i]$ is represented by

$$\cdots\to 0\to 0\to \ker(C^i\to C^{i+1})/\operatorname{im}(C^{i-1})\to 0\to 0\to\cdots$$

and we get a map of complexes with $\ker(C^i\to C^{i+1})/\operatorname{im}(C^{i-1})$ going to $C^i/\operatorname{im}(C^{i-1})$ which you can check is an isomorphism in cohomology (the first complex has no cohomology except in the $C^i/\operatorname{im}C^{i-1}$ term, where the cohomology is the kernel of the map out of it, which is the same as the second complex). So we get a distinguished triangle

$$\tau^{<i}C\to C\to H^i(C)[-i]\to$$

and therefore a long exact sequence

$$\cdots\to\mathrm{Hom}(H^i(C)[-1],D)\to\mathrm{Hom}(C,D)\to\mathrm{Hom}(\tau^{<i}C,D)\to\cdots.$$

It suffices to know the left and right terms above are finitely generated, because then the image of the middle term in the right term is also finitely generated since A is noetherian, and therefore the middle term is finitely generated. The left term has cohomology in 1 degree, and the right term has cohomology in $N-1$ degrees, so we can induct on N. Arguing similarly for D, we reduce to the case that $C=X[-i]$ and $D=Y[-j]$ where $X,Y\in A-Mod$. Then

$$\mathrm{Hom}_{D^b(A)}(C,D)=\mathrm{Ext}_A^{i-j}(X,Y)$$

which is a finitely generated A-module (take a projective resolution of X; then the Ext-terms are finite projective modules over A).

2. We claim that if $B\xrightarrow{f}C\xrightarrow{g}D\to$ is a distinguished triangle in $D^b(A)$, and we have two morphisms of distinguished triangles from $B\xrightarrow{f}C\xrightarrow{g}D\to$ to itself given by $0:B\to B$, $\gamma,\gamma':C\to C$, and $0:D\to D$, then $\gamma'\circ\gamma=0$. (If this were a statement about modules, we would just be saying that if you have a one-step filtration of a module C, and a morphism from the module to itself that maps the whole module to

the filtered piece B and the filtered piece to 0, then if you compose the two maps you get 0, which is obvious.) Proof: let $\gamma \in \mathrm{Hom}(C, C)$. We have an exact sequence

$$\cdots \to \mathrm{Hom}(C, B) \to \mathrm{Hom}(C, C) \to \mathrm{Hom}(C, D) \to \cdots$$

taking γ to $0 \in \mathrm{Hom}(C, D)$, so γ comes from some $\beta \in \mathrm{Hom}(C, B)$, that is, $f \circ \beta = \gamma$. Similarly, we have an exact sequence

$$\cdots \to \mathrm{Hom}(D, C) \to \mathrm{Hom}(C, C) \to \mathrm{Hom}(B, C) \to \cdots$$

taking γ' to $0 \in \mathrm{Hom}(B, C)$, so γ' comes from some $\delta' \in \mathrm{Hom}(D, C)$, that is, $\gamma' = \delta' \circ g$. Then

$$\gamma' \circ \gamma = \delta' \circ g \circ f \circ \beta = \delta' \circ 0 \circ \beta = 0$$

since $B \xrightarrow{f} C \xrightarrow{g} D \to$ is an exact triangle.

Now we can prove the lemma by inducting from this claim. Choose i maximal such that $H^i(C) \neq (0)$. Again we have a distinguished triangle

$$\tau^{<i} C \to C \to H^i(C)[-i] \to$$

and also a natural map

$$\mathrm{End}_{D^b(A)}(C) \to \mathrm{End}_{D^b(A)}(\tau^{<i} C) \oplus \mathrm{End}_A(H^i(C))$$

and we see from the claim that

$$\ker \left(\mathrm{End}_{D^b(A)}(C) \to \mathrm{End}_{D^b(A)}(\tau^{<i} C) \oplus \mathrm{End}_A(H^i(C))\right)^2 = (0).$$

$\mathrm{End}_A(H^i(C))$ is the part we cared about, so we can continue inductively with $\tau^{<i} C$. We conclude that

$$\ker \left(\mathrm{End}_{D^b(A)}(C) \to \bigoplus_i \mathrm{End}_A(H^i(C))\right)^{2^{N-1}} = (0).$$

This is weaker than the stated bound in the lemma but it suffices for all applications. If you really want to replace 2^{N-1} by $N-1$, you split off the cohomology in the middle instead of from the top, i.e. by looking at

$$\tau^{<i} C \to C \to \tau^{\geq i} C \to$$

where i is chosen so that $\tau^{<i} C$ and $\tau^{\geq i} C$ have cohomology in at most $\lceil N/2 \rceil$ degrees. $\square$

There can be a lot of torsion in these cohomology groups and Hecke algebras, unlike the ones for modular forms.

7.3 Galois representations

Theorem 7.3.1 (Scholze). *1. Suppose $\mathfrak{m}$ is a maximal idea in $\mathbb{T}^S(U, \underline{\lambda})$. Then there is a continuous semisimple representation*

$$\overline{r}_{\mathfrak{m}} : G_F \to GL_n(\mathbb{T}^S(U, \lambda)/\mathfrak{m})$$

that is unramified outside $S \cup \{w|l\}$, such that if v is not in this set of bad primes, then $\overline{r}_{\mathfrak{m}}(\mathrm{Frob}_v)$ has characteristic polynomial $P_v(X)$ (so we've described the Frobenius semisimplification of the representation at v). Moreover, $\overline{r}_{\mathfrak{m}}$ is determined up to conjugation.

2. Suppose that $\mathfrak{m}$ is non-Eisenstein (see definition below). Then there is an integer $N = N([F : \mathbb{Q}], n)$ only depending on $[F : \mathbb{Q}]$ and n, an ideal I of $\mathbb{T}^S(U, \underline{\lambda})_{\mathfrak{m}}$ with $I^N = (0)$, and a continuous representation

$$r_{\mathfrak{m}} : G_F \to GL_n(\mathbb{T}^S(U, \lambda)_{\mathfrak{m}}/I)$$

such that $r_{\mathfrak{m}}$ is unramified away from $S \cup \{w|l\}$ and if $v \notin S \cup \{w|l\}$ then $r_{\mathfrak{m}}(\mathrm{Frob}_v)$ has characteristic polynomial $P_v(X)$. (Again this determines $r_{\mathfrak{m}}$ up to conjugation.)

Definition 7.3.2. We will call $\mathfrak{m}$ non-Eisenstein if $\overline{r}_{\mathfrak{m}}$ is absolutely irreducible (irreducible over the algebraic closure of the residue field).

It's tempting to conjecture that we can take $I = 0$. We don't know how to do that, but in practice it seems harmless due to the lack of dependence of N on U, λ, etc.

We will need some refinements of this. Let $S \supset R \coprod T$ such that $R^c = R$, if $v \in R$ and $v|p$ then p splits in F_0 and $p \neq l$, $T^c = T$, and if $v \in T$ then $v|l$. Let

$$U = U_S \times \prod_{v \notin S} GL_n(\mathcal{O}_{F,v}) \subset U' = U'_S \times \prod_{v \notin S} GL_n(\mathcal{O}_{F,v})$$

be neat open compact subgroups such that U is normal in U' and U'/U is abelian. Assume that

$$U_S = U_{S-R-T} \times \prod_{v \in R \cup T} U_v, \qquad U'_S = U'_{S-R-T} \times \prod_{v \in R \cup T} U'_v.$$

For $v \in R$, assume

$$\mathrm{Iw}_v \supset U'_v \supset U_v \supset \mathrm{Iw}_{v,1}$$

where

$$\mathrm{Iw}_v = \{g \in GL_n(\mathcal{O}_{F,v}) \mid g \pmod v \text{ is upper triangular}\}$$

contains

$$\mathrm{Iw}_{v,1} = \{g \in \mathrm{Iw}_v \mid g \pmod v \text{ has all diagonal entries } 1\}.$$

Note that

$$\mathrm{Iw}_v / \mathrm{Iw}_{v,1} \xrightarrow{\sim} (k(v)^\times)^n$$

with the isomorphism sending g to the diagonal entries of $g \pmod v$. Let

$$\Xi_v = (F_v^\times)^n / \ker((\mathcal{O}_{F,v}^\times)^n \to (k(v)^\times)^n)$$

(so $\Xi_v \cong \mathbb{Z}^n \times (k(v)^\times)^n$); this contains the semigroup

$$\Xi_v^+ = \{\underline{\alpha} \in \Xi_v \mid v(\alpha_1) \geq \cdots \geq v(\alpha_n)\}.$$

Let

$$\begin{aligned} \mathbb{Z}[\Xi_v^+] &\to \mathcal{O}[U_v \backslash GL_n(F_v)/U_v] \\ (\alpha_1, \ldots, \alpha_n) \in \Xi_v^+ &\mapsto [U_v \operatorname{diag}(\alpha_1, \ldots, \alpha_n) U_v] q_v^{(n-1)v(\alpha_1)+(n-2)v(\alpha_2)+\cdots+v(\alpha_{n-1})}. \end{aligned}$$

Theorem: $\mathbb{Z}[\Xi_v^+] \hookrightarrow \mathbb{Z}[\Xi_v]$ extends uniquely to $\mathcal{O}[U_v \backslash GL_n(F_v)/U_v] \to \mathbb{Z}[\Xi_v]$ (but not using the above formula! Instead you have to actually write the element of Ξ_v as the difference of two elements of Ξ_v^+ and take their quotient in the image.)

8 April 22: setup for generalizing Scholze's theorem.

8.1 Subgroups

Last time, we stated Scholze's theorem giving the existence of Galois representations associated to the Hecke algebra $\mathbb{T}^S(U, \lambda)$ with coefficients in the locally constant sheaf of $\mathcal{O}$-modules $\mathscr{F}_\lambda$ coming from the representation ρ_λ of $G = \operatorname{res}_{\mathbb{Z}}^{\mathcal{O}_F} GL_n$ associated to the weight $\lambda \in (\mathbb{Z}_+^n)^{\operatorname{Hom}(F,L)}$.

Today we will state a refinement with $S \supset Q \coprod R \coprod T$ (we tried to get away without Q last time, handling it simultaneously with R, but we're giving up on that), where $Q^c = Q$, $R^c = R$, $T^c = T$. If $v \in Q \coprod R$ with $v|p$, then we assume that $p \neq l$ and p splits in F_0. If $v \in T$, we assume that $v|l$ (recall we are assuming that l is split in F_0, so v is split over F^+). We will choose compact open subgroups

$$U = U_S \times \prod_{v \notin S} GL_n(\mathcal{O}_{F,v}) \trianglelefteq U' = U'_S \times \prod_{v \notin S} GL_n(\mathcal{O}_{F,v})$$

of the form

$$\begin{aligned} U_S &= U_{S-(Q \cup R \cup T)} \times \prod_{v \in Q} U_v \times \prod_{v \in R} U_v \times \prod_{v \in T} U_v \\ U'_S &= U_{S-(Q \cup R \cup T)} \times \prod_{v \in Q} U'_v \times \prod_{v \in R} U_v \times \prod_{v \in T} U'_v. \end{aligned}$$

That is, we assume that U'_S can be bigger at the places in Q and T, and is the same elsewhere. We will choose these as follows. For $v \in Q$, let

$$U'_v = \mathrm{Iw}_v = \{g \in GL_n(\mathcal{O}_{F,v}) \mid g \pmod{v} \text{ is upper triangular}\}$$

and choose $U_v \subset U'_v$ to contain

$$\mathrm{Iw}_{v,1} = \{g \in \mathrm{Iw}_v \mid \text{ the diagonal entries of } g \pmod{v} \text{ are all } 1\}.$$

Recall that we have an isomorphism

$$\begin{aligned} \mathrm{Iw}_v / \mathrm{Iw}_{v,1} &\xrightarrow{\sim} (k(v)^\times)^n \\ (a_{ij}) &\mapsto (a_{11} \pmod{v}, \ldots, a_{nn} \pmod{v}) \end{aligned}$$

and that we introduced the group

$$\Xi_v = (F_v^\times)^n / \ker((\mathcal{O}_{F,v}^\times)^n \to (k(v)^\times)^n)$$

containing the semigroup

$$\Xi_v^+ = \{(\alpha_1, \ldots, \alpha_n) \mid v(\alpha_1) \geq v(\alpha_2) \geq \cdots \geq v(\alpha_n)\}.$$

Recall that if $q_v = \#k(v)$, we have a map

$$\begin{aligned} \mathbb{Z}[\Xi_v^+] &\to \mathbb{Z}[q_v^{-1}][U_v \backslash GL_n(F_v)/U_v] \\ \underline{\alpha} &\mapsto [U_v \operatorname{diag}(\alpha_1, \ldots, \alpha_n) U_v] q_v^{(n-1)v(\alpha_1) + \cdots + v(\alpha_{n-1})} \end{aligned}$$

which extends uniquely to $\mathbb{Z}[\Xi_v]$; that is, the elements we wrote above on the right are invertible in the Hecke algebra. Furthermore, looking at the natural action of S_n on Ξ_v by permuting the α_is, this map takes

$$\mathbb{Z}[\Xi_v]^{S_n} \to Z(\mathbb{Z}[q_v^{-1}][U_v \backslash GL_n(F_v)/U_v])$$

(the whole Hecke algebra is not generally abelian). For $\alpha \in F_v^\times$, let $t_{v,i,\alpha} \in \mathbb{Z}[\Xi_v]$ be the element $(1, \ldots, 1, \alpha, 1, \ldots, 1)$ where α is in the ith location. Let

$$P_{v,\alpha}(x) = \prod_{i=1}^n (X - t_{v,i,\alpha}) \in \mathbb{Z}[\Xi_v]^{S_n}[X].$$

We have a map $U_v \to (k(v)^\times)^n \hookrightarrow \Xi_v$.

Lemma 8.1.1. *If π is a smooth irreducible representation of $GL_n(F_v)$ over $\mathbb{C}$, then TFAE:*

- $\pi^{U_v} \neq (0)$.
- *π is a subquotient of $n-\operatorname{Ind}_{B_n(F_v)}^{GL_n(F_v)} \psi$ where ψ factors as $B_n(F_v) \to (F_v^\times)^n \to \Xi_v/$(image of U_v).*
- *$\operatorname{rec}(\pi|\cdot|^{(1-n)/2})^{ss} \cong \varphi_1 \oplus \cdots \oplus \varphi_n$, where the φ_i are tamely ramified characters such that $(\varphi_1, \ldots, \varphi_n) \circ \operatorname{art}_{F_v}$ factors through $\Xi_v / \operatorname{im}(U_v)$.*

In this case, $Z(\mathbb{C}[U_v \backslash GL_n(F_v)/U_v])$ acts by scalars on π^{U_v} (by Schur's lemma, because the entire Hecke algebra acts irreducibly), say via

$$\theta_\pi : Z(\mathbb{C}[U_v \backslash GL_n(F_v)/U_v]) \to \mathbb{C},$$

and for all $\alpha \in F_v^\times$, $\operatorname{rec}(\pi|\cdot|^{(1-n)/2})(\operatorname{art}_{F_v}(\alpha))$ has characteristic polynomial $\theta_\pi(P_{v,\alpha}(x))$.

Now let $v \in R$, so $U'_v = U_v = \operatorname{Iw}_v$. Choose $\chi_v : U_v / \operatorname{Iw}_{v,1} \cong (k(v)^\times)^n \to \mathcal{O}^\times$. Let

$$P_{v,\alpha}(x) = \prod_{i=1}^n (X - \chi_{v,i}^{-1}(\alpha)) \in \mathcal{O}[X].$$

Lemma 8.1.2. *If π is an irreducible smooth representation of $GL_n(F_v)$ over $\mathbb{C}$, then TFAE:*

- $\pi^{\mathrm{Iw}_v,\chi_v^{-1}} \neq (0)$.
- *π is a subquotient of* $n-\mathrm{Ind}_{B_n(F_v)}^{GL_n(F_v)}\psi$ *where* $\psi|_{(\mathcal{O}_{F,v}^\times)^n} = \chi_v^{-1}$.
- $\mathrm{rec}(\pi|\cdot|^{(1-n)/2})^{ss}|_{I_{F_v}} = \bigoplus \chi_{v,i}^{-1} \circ \mathrm{art}_{F_v}^{-1}$.

(This lemma is the same as the one for primes in Q, except we are not introducing the Hecke algebra, and just fixing a character and keeping track of what's happening with the inertia. It can be easily deduced from the lemma for primes in Q.)

Finally, let $v \in T$. In this case, choose $b_v, c_v \in \mathbb{Z}_{\geq 0}$ with $b_v \leq c_v$, and let

$$\begin{aligned} U_v = \mathrm{Iw}_v(b_v, c_v) = \{g \in GL_n(\mathcal{O}_{F,v}) \mid & g \pmod{v^{c_v}} \text{ is upper triangular} \\ & g \pmod{v^{b_v}} \text{ has diagonal entries equal to } 1\}. \end{aligned}$$

Let $U_v' = \mathrm{Iw}_v(0, c_v) \supset U_v$. Let π_v be a uniformizer in $\mathcal{O}_{F,v}$ and

$$\Delta_v = \coprod_{\mu \in \mathbb{Z}_+^n} \mathrm{Iw}_v \,\mathrm{diag}(\pi_v^{\mu_1}, \ldots, \pi_v^{\mu_n})\, \mathrm{Iw}_v\,.$$

This is a semigroup which acts on V_λ. We should be careful about the action because just multiplying a vector in V_λ by an element of Δ_v might change the lattice, so we have to rescale the multiplication. If $g \in \Delta_v$, say $g \in \mathrm{Iw}_v \,\mathrm{diag}(\pi_v^{\mu_1}, \ldots, \pi_v^{\mu_n})\, \mathrm{Iw}_v$, then we define

$$g.x = \prod_{i=1}^n \left(\prod_{\tau \in \mathrm{Hom}_{\mathbb{Q}_l}(F_v, L)} \tau(\pi_v)^{\lambda_{\tau,n+1-i}} \right)^{-\mu_i} gx.$$

This really does preserve $V_\lambda \subset V_\lambda \otimes_{\mathbb{Z}} \mathbb{Q}_l$. Note that this . action depends on the choice of π_v.

Let

$$U_{v,i} = [U_v \,\mathrm{diag}(\pi_v, \ldots, \pi_v, 1, \ldots, 1) U_v] \in \mathbb{Z}[U_v \backslash \Delta_v / U_v]$$

with i copies of π_v on the diagonal (we apologize for using $U_{v,i}$ to mean the Hecke operator but U_v to mean the open compact subgroup). Let

$$U_v^+ = [U_v \,\mathrm{diag}(\pi_v^{n-1}, \pi_v^{n-2}, \ldots, \pi_v, 1) U_v] = U_{v,1} \cdots U_{v,n-1}.$$

For $\alpha \in (\mathcal{O}_{F,v}^\times)^n$, we have the associated diamond operator

$$\langle \alpha \rangle = [U_v \,\mathrm{diag}(\alpha_1, \ldots, \alpha_n) U_v].$$

Let

$$\mathscr{H}_v = \mathbb{Z}[U_{v,1}, \ldots, U_{v,n}, U_{v,n}^{-1}, \langle \alpha \rangle]_{\alpha \in (\mathcal{O}_{F,v}^\times)^n} \subset \mathbb{Z}[U_v \backslash \Delta_v / U_v].$$

This is commutative, unlike $\mathbb{Z}[U_v \backslash \Delta_v / U_v]$, which is very complicated.

Let

$$\rho = \rho_\lambda \otimes \bigotimes_{v \in R} \chi_v$$

where ρ_λ is a representation of $GL_n(\mathcal{O}_{F,l})$, and χ_v is the previously chosen character of Iw_v for $v \in R$. Write $\mathscr{F}_\rho = \mathscr{F}_{\lambda,\underline{\chi}}$. Let

$$R\Gamma(X_U, \mathscr{F}_{\lambda,\underline{\chi}})_{U'} = R\Gamma(X_{U'}, \pi_{U/U',*}\mathscr{F}_{\lambda,\underline{\chi}}) \in D(\mathcal{O}[U'/U]).$$

This has a Hecke action. Let $\mathbb{T}^S_{Q,T}(U, \lambda, \underline{\chi})$ be the $\mathcal{O}$-algebra in $\mathrm{End}_{D(\mathcal{O})}(R\Gamma(X_U, \mathscr{F}_{\lambda,\underline{\chi}}))$ generated by the image of

$$\mathscr{H}^S \otimes \bigotimes_{v\in Q} \mathbb{Z}[\Xi_v] \otimes \bigotimes_{v\in T} \mathscr{H}_v$$

But if we want to work in $D(\mathcal{O}[U'/U])$, maybe we really want $\mathbb{T}^S_{Q,T}(U'/U, \lambda, \underline{\chi})$, the $\mathcal{O}[U'/U]$-algebra in $\mathrm{End}_{D(\mathcal{O}[U'/U])}(R\Gamma(X_U, \mathscr{F}_{\lambda,\chi'})_{U'})$ generated by etc. But the thing is that we never actually defined the Hecke actions on $R\Gamma(X_U, \mathscr{F}_\rho)_{U'}$, and this definition is non-obvious, so now we have to go back and do that.

8.2 Going between U and U'

Let $\Delta \supset U' \trianglerighteq U$ where Δ is a semigroup and U', U are neat open compact subgroups. Suppose that these satisfy

a. if $h \in \Delta$ then $U' = U(U' \cap h^{-1}U'h)$.

b. if $h \in \Delta$ then $U' \cap hUh^{-1} \supset U$.

c. if $h \in \Delta$ and $u \in U' \cap h^{-1}U'h$, then the quotient of the elements $huh^{-1}, u^{-1} \in U'$ lies in U, that is, $(huh^{-1})u^{-1} \in U$. In particular, since h can be chosen in U', this implies that U'/U is abelian.

Remark 1. If $h \in \Delta$ satisfies these three properties, then every element of $U'hU'$ satisfies them too.

Example 8.2.1. If $v \in Q$, the semigroup

$$\Delta_v = \coprod_{v(\alpha_1)\geq\cdots\geq v(\alpha_n)} \mathrm{Iw}_v \operatorname{diag}(\alpha_1, \ldots, \alpha_n) \mathrm{Iw}_v$$

together with U'_v, U_v satisfies the three properties. Why? We just need to check this for $h = \operatorname{diag}(\alpha_1, \ldots, \alpha_n)$, where $v(\alpha_1) \geq \cdots \geq v(\alpha_n)$. First, what is $\mathrm{Iw}_v \cap h^{-1} \mathrm{Iw}_v h$? Conjugating by h multiplies the ijth entry by α_j/α_i. So in an element of $\mathrm{Iw}_v \cap h^{-1} \mathrm{Iw}_v h$, the diagonal entries can be any units, the above-diagonal entries can also be any units, and the ijth below-diagonal entry has to be divisible by $v^{1+v(\alpha_j)-v(\alpha_i)}$. So indeed

$$\mathrm{Iw}_v = \mathrm{Iw}_{v,1}(\mathrm{Iw}_v \cap h^{-1} \mathrm{Iw}_v h)$$

and if $u \in \mathrm{Iw}_v$ then $huh^{-1}u^{-1} \in \mathrm{Iw}_{v,1}$ (since the diagonal entries of huh^{-1} and u are the same), giving parts a and c. For part b, in an element of $h \mathrm{Iw}_{v,1} h^{-1} \cap \mathrm{Iw}_v$, the diagonal entries can be any 1 (mod v) units, the below-diagonal entries are $\equiv 0 \pmod v$, and the ijth above-diagonal entry is $\equiv 0 \pmod{v^{v(\alpha_i)-v(\alpha_j)}}$. This is indeed in $\mathrm{Iw}_{v,1}$.

The same is true for $v \in T$.

Assume these properties and let ρ be a representation of Δ. Then $\mathbb{Z}[U\backslash\Delta/U]$ acts on $R\Gamma(X_U, \mathscr{F}_\rho)_{U'} \in D(\mathcal{O}[U'/U])$ compatibly with the $\mathbb{Z}[U\backslash\Delta/U]$-action on $R\Gamma(X_U, \mathscr{F}_\rho) \in D(\mathcal{O})$ and the forgetful functor $D(\mathcal{O}[U'/U]) \to D(\mathcal{O})$. The idea is as follows. By definition

$$R\Gamma(X_{U'}, \pi_{U/U',*}\mathscr{F}_\rho) = R\Gamma\left(X_{U'}, \mathscr{F}_{\mathrm{Ind}_U^{U'}\rho}\right).$$

U/U' acts on $\pi_{U/U',*}\mathscr{F}_\rho$. $\mathrm{Ind}_U^{U'}\rho$ has an action of U' and a commuting action of U'/U, as follows. If $\varphi \in \mathrm{Ind}_U^{U'}\rho$, so $\varphi : U' \to F_\rho$ is a map satisfying $\varphi(uv) = u\varphi(v)$ for all $u \in U$ and $v \in U'$, then $w \in U'$ acts by $(w\varphi)(v) = \varphi(vw)$ and $w \in U'/U$ acts by $(w.\varphi)(v) = w\varphi(w^{-1}v)$ (you can check this only depends on the coset wU).

Now we will define the action of UgU on this. We have

$$g^* : R\Gamma\left(X_{U'}, \mathscr{F}_{\mathrm{Ind}_U^{U'}\rho}\right) \to R\Gamma\left(X_{U'\cap gU'g^{-1}}, g^*\mathscr{F}_{\mathrm{Ind}_U^{U'}\rho}\right)$$

and using the fact that $U' = U(g^{-1}U'g \cap U')$, the target is isomorphic to

$$\begin{aligned} R\Gamma\left(X_{U'\cap gU'g^{-1}}, \mathscr{F}_{\mathrm{Ind}_{U'\cap gUg^{-1}}^{U'\cap gU'g^{-1}}(\rho\circ\mathrm{conj}_{g^{-1}})}\right) &\cong R\Gamma\left(X_{U'}, \mathscr{F}_{\mathrm{Ind}_{U'\cap gU'g^{-1}}^{U'}\mathrm{Ind}_{U'\cap gUg^{-1}}^{U'\cap gU'g^{-1}}(\rho\circ\mathrm{conj}_{g^{-1}})}\right) \\ &\cong R\Gamma\left(X_{U'}, \mathscr{F}_{\mathrm{Ind}_{U'\cap gUg^{-1}}^{U'}(\rho\circ\mathrm{conj}_{g^{-1}})}\right) \end{aligned}$$

which is taken by T_g to $R\Gamma\left(X_{U'}, \mathscr{F}_{\mathrm{Ind}_U^{U'}\rho}\right)$, where

$$T_g : \mathrm{Ind}_{U'\cap gUg^{-1}}^{U'}(\rho\circ\mathrm{conj}_{g^{-1}}) \to \mathrm{Ind}_U^{U'}\rho$$

is given by

$$(T_g\varphi)(v) = \sum_{u_i \in U'\cap gUg^{-1}\backslash U} u_i^{-1}g\varphi(u_iv)$$

for all $v \in U'$ (using $U' \cap gUg^{-1} \subset U$). One must check that these maps preserve the $\mathcal{O}[U'/U]$-actions; this is because using $U' = U(g^{-1}U'g \cap U')$, the embedding

$$\mathcal{O}[g^{-1}U'g \cap U'/g^{-1}U'g \cap U] \xrightarrow{\sim} \mathcal{O}[U'/U]$$

becomes an isomorphism. One must also check that the described action only depends on UgU, and is compatible with composition:

$$[UgU][UhU] = \sum [Uk_iU].$$

We are not going to do these in class. Anyway, now we can define $\mathbb{T}_{Q,T}^S(U'/U, \lambda, \underline{\chi})$ to be the $\mathcal{O}[U'/U]$-subalgebra of $\mathrm{End}_{\mathcal{O}[U'/U]}(R\Gamma(X_U, \mathscr{F}_{\lambda,\underline{\chi}})_{U'})$ generated by

$$\mathscr{H}^S \otimes \bigotimes_{v\in Q}\mathbb{Z}[\Xi_v^+] \otimes \bigotimes_{v\in T}\mathscr{H}_v$$

(note we didn't check the previous conditions for Ξ_v, only Ξ_v^+). This is a commutative $\mathcal{O}[U'/U]$-algebra which is finitely generated as an $\mathcal{O}$-module. We have a map

$$\mathbb{T}^S_{Q,T}(U'/U, \lambda, \underline{\chi}) \to \mathbb{T}^S_{Q,T}(U, \lambda, \underline{\chi}).$$

To make sure this map exists, we need to check that the U'/U-action is recovered from the Ξ_v^+-action for $v \in Q$ and the $\langle \alpha \rangle$-action for $v \in T$. Note that the kernel is nilpotent of exponent bounded in terms of only n and $[F : \mathbb{Q}]$. This is because both $\mathbb{T}^S_{Q,T}(U'/U, \lambda, \underline{\chi})$ and $\mathbb{T}^S_{Q,T}(U, \lambda, \underline{\chi})$ act on $\bigoplus H^i(X_U, \mathscr{F}_{\lambda,\underline{\chi}})$, because if $C \in D(\mathcal{O}[U'/U])$ then $H^i(C) \cong H^i((\text{forget } U'/U\text{-action})C) \in D(\mathcal{O})$. So any element of $\mathbb{T}^S_{Q,T}(U'/U, \lambda, \underline{\chi})$ that goes to 0 in $\mathbb{T}^S_{Q,T}(U, \lambda, \underline{\chi})$ must act trivially on $\bigoplus H^i(X_U, \mathscr{F}_{\lambda,\underline{\chi}})$, but we saw last time that the kernel of $\mathbb{T}^S_{Q,T}(U'/U, \lambda, \underline{\chi})$ in the endomorphism ring of $\bigoplus H^i(X_U, \mathscr{F}_{\lambda,\underline{\chi}})$ is indeed nilpotent of bounded exponent as described.

Next time, we will restate/extend Scholze in this more general context.

9 April 27: extending Scholze's theorem.

9.1 Recap

Recall: F_0 is an imaginary quadratic field, F^+ is totally real, and $F = F_0F^+$. $G = \operatorname{res}^{\mathcal{O}_F}_{\mathbb{Z}} GL_n$, $G(\mathbb{Q}) = GL_n(F)$, X the symmetric space for G of real dimension $[F^+ : \mathbb{Q}]n^2 - 1$. l is split in F_0, $L/\mathbb{Q}_l$ is a finite extension with ring of integers $\mathcal{O}$ and residue field $\mathbb{F} = \mathcal{O}/\lambda$, sufficiently large to contain all embeddings of F into $\overline{L}$. For $\lambda \in (\mathbb{Z}^n_+)^{\operatorname{Hom}(F,L)}$, V_λ is the corresponding algebraic representation of G defined over $\mathcal{O}$. S is a finite set of bad places of F satisfying some conditions.

$$U = \prod_{v \in S} U_v \times \prod_{v \notin S} GL_n(\mathcal{O}_{F,v}) \trianglelefteq U' = \prod_{v \in S} U'_v \times \prod_{v \notin S} GL_n(\mathcal{O}_{F,v})$$

are neat open compact subgroups. S contains $Q \coprod R \coprod T$ satisfying some conditions; places in Q and R don't divide l, places in T do. We have $U'_v = \mathrm{Iw}_v \supset U_v \supset \mathrm{Iw}_{v,1}$ for $v \in Q$, $U'_v = U_v = \mathrm{Iw}_v$ for $v \in R$, and $U'_v = \mathrm{Iw}_v(0, c_v) \supset U_v = \mathrm{Iw}_v(b_v, c_v)$ for $v \in T$ for some $b_v \leq c_v$.

$$\mathscr{H} = \mathscr{H}^S = \bigotimes_{v \notin S}{}' \mathbb{Z}[GL_n(\mathcal{O}_{F,v} \backslash GL_n(F_v)/GL_n(\mathcal{O}_{F,v})] \otimes \bigotimes_{v \in Q} \mathbb{Z}[\Xi_v] \otimes \bigotimes_{v \in T} \mathscr{H}_v$$

where

$$\Xi_v = (F_v^\times)^n / \ker(\mathcal{O}^\times_{F,v} \to k(v)^\times)^n \supset \Xi_v^+ = \{\underline{\alpha} \in \Xi_v \mid v(\alpha_1) \geq \cdots \geq v(\alpha_n)\}$$

where we had a map

$$\Xi_v^+ \to \mathcal{O}[q_v^{-1}][U_v \backslash \Delta_v / U_v].$$

For $v \in R$, we chose

$$\chi_v : (k(v)^\times)^n = \mathrm{Iw}_v / \mathrm{Iw}_{v,1} \to \mathcal{O}^\times$$

and wrote $\chi = \otimes_{v\in R}\chi_v$. We set

$$\rho = \rho_\lambda \otimes \bigotimes_{v\in R} \chi_v : U \to V_\lambda(\mathcal{O})$$

and wrote $\mathscr{F}_{\lambda,\underline{\chi}} = \mathscr{F}_\rho$. We considered

$$R\Gamma(X_U, \mathscr{F}_{\lambda,\underline{\chi}})_{U'} \in D(\mathcal{O}[U'/U])$$

which has an action of $\mathscr{H}$, hence of $\mathbb{T}^S_{Q,T}(U'/U, \lambda, \underline{\chi})$, the $\mathcal{O}[U'/U]$-algebra generated by $\mathscr{H}$ in the endomorphism ring of $R\Gamma$. $\mathbb{T}^S_{Q,T}(U'/U, \lambda, \underline{\chi})$ is a commutative $\mathcal{O}[U'/U]$-algebra, finitely generated as a module over $\mathcal{O}$. It surjects onto $\mathbb{T}^S_{Q,T}(U, \lambda, \underline{\chi})$, the corresponding algebra for $D(\mathcal{O})$ forgetting the U'/U-action, which maps to

$$\mathrm{End}_{\mathcal{O}}\left(\bigoplus_i H^i(X_U, \mathscr{F}_{\lambda,\underline{\chi}})\right).$$

We saw that the composite map has nilpotent kernel with degree bounded depending only on n and $[F : \mathbb{Q}]$.

9.2 More Hecke algebras

Fact: for $v \in Q$, the map $\mathbb{Z}[\Xi_v^+] \to \mathbb{T}^S_{Q,T}(U'/U, \lambda, \underline{\chi})$ extends uniquely to a map $\mathbb{Z}[\Xi_v] \to \mathbb{T}^S_{Q,T}(U'/U, \lambda, \underline{\chi})$. To show this, we need to show that elements of Ξ_v^+ map to units in $\mathbb{T}^S_{Q,T}(U'/U, \lambda, \underline{\chi})$, or even just in $\mathbb{T}^S_{Q,T}(U, \lambda, \underline{\chi})$, since the map from the former to the latter has nilpotent kernel. The map $\mathbb{Z}[\Xi_v^+] \to \mathrm{End}_{D(\mathcal{O})}(R\Gamma(X_U, \mathscr{F}_{\lambda,\underline{\chi}}))$ extends to $\mathbb{Z}[\Xi_v]$, because

$$\mathbb{Z}[\Xi_v^+] \to \mathbb{Z}[q_v^{-1}][U_v\backslash GL_n(F_v)/U_v]$$

extends to $\mathbb{Z}[\Xi_v]$, and $\mathbb{Z}[q_v^{-1}][U_v\backslash GL_n(F_v)/U_v]$ acts on $R\Gamma$.

Let $\tilde{\mathbb{T}}^S_{Q,T}(U, \lambda, \underline{\chi})$ be the $\mathbb{T}^S_{Q,T}(U, \lambda, \underline{\chi})$-algebra generated by the image of $\mathbb{Z}[\Xi_v]$ in $\mathrm{End}_{D(\mathcal{O})}(R\Gamma(X_U, \mathscr{F}_{\lambda,\underline{\chi}}))$. Then $\tilde{\mathbb{T}}^S_{Q,T}(U, \lambda, \underline{\chi})$ is finite as an $\mathcal{O}$-module, hence finite as a $\mathbb{T}^S_{Q,T}(U, \lambda, \underline{\chi})$-module. But it is a general fact that if $A \subset B$ is an integral extension and $a \in A$ has an inverse in B, then it has an inverse in A. (Proof: if $b \in B$ is an inverse to a and if

$$b^n + c_{n-1}b^{n-1} + \cdots + c_0 = 0$$

with $c_i \in A$, then multiplying through by a^n we get $1 + c_{n-1}a + \cdots + c_0a^n = 0$, i.e.

$$a(-c_{n-1} - c_{n-2}a - \cdots - c_0a^{n-1}) = 1,$$

and the expression in the parentheses is in A.) Since finite extensions are integral, and elements of Ξ_v^+ have inverses in $\tilde{\mathbb{T}}^S_{Q,T}(U, \lambda, \underline{\chi})$, they also have inverses in $\mathbb{T}^S_{Q,T}(U, \lambda, \underline{\chi})$.

(If we'd been just working in $D(\mathcal{O})$, we could have worked with $\mathbb{Z}[\Xi_v]$ from the beginning. But in $D(\mathcal{O}[U'/U])$ it seems difficult to directly define the action of $\mathbb{Z}[\Xi_v]$, so we had to just define the action of $\mathbb{Z}[\Xi_v^+]$ and then abstractly extend it in this way.)

We have

$$\mathbb{T}^S_{Q,T}(U'/U,\lambda,\underline{\chi}) = \prod_{\mathfrak{m}\text{ maximal ideal}} \mathbb{T}^S_{Q,T}(U'/U,\lambda,\underline{\chi})_{\mathfrak{m}}$$

since the LHS is a finite module over $\mathcal{O}$, hence semilocal. We write $1 = \sum_{\mathfrak{m}} e_{\mathfrak{m}}$ where $e_{\mathfrak{m}}$ is the idempotent that is 1 in the $\mathfrak{m}$-component and 0 elsewhere. We have $e_{\mathfrak{m}}^2 = e_{\mathfrak{m}}$ and $e_{\mathfrak{m}}e_{\mathfrak{n}} = 0$ if $\mathfrak{m} \neq \mathfrak{n}$. We can write

$$R\Gamma(X_U, \mathscr{F}_{\lambda,\underline{\chi}})_{U'} = \bigoplus_{\mathfrak{m}} e_{\mathfrak{m}} R\Gamma(X_U, \mathscr{F}_{\lambda,\underline{\chi}})_{U'}$$

in $D(\mathcal{O}[U'/U])$, even though it's not obvious what this means because $D(\mathcal{O}[U'/U])$ is not an abelian category and does not have images. So $e_{\mathfrak{m}} R\Gamma(X_U, \mathscr{F}_{\lambda,\underline{\chi}})_{U'}$ is not defined as an image. However, fact: if A is a ring, then

i. $A-Mod$ has small direct sums (in particular countably infinite direct sums, which is not guaranteed for a general abelian category), and these preserve short exact sequences.

ii. $D(A)$ has small direct sums and these preserve exact triangles.

Moreover, any triangulated category satisfying (ii) has "images" of idempotents. So if $C \in D(A)$ and $e \in \mathrm{End}_{D(A)}(C)$ with $e^2 = e$, then we have a decomposition $C = eC \oplus (1-e)C$ where $e = 1$ on eC and $e = 0$ on $(1-e)C$, and this decomposition is unique. This is a theorem of Bökstedt and Neeman, Compositio Math 1993 [2]. The same is true for $D^b(A)$ instead of $D(A)$. This doesn't follow directly from Bökstedt and Neeman because $D^b(A)$ does not have infinite direct sums, but you can conclude it from the statement for $D(A)$ by general homological algebra stuff.

Let $U'' = \prod_{v\in R} \mathrm{Iw}_{v,1} \times U^R \subset U \subset U'$; we have $U'' \trianglelefteq U'$. We have

$$R\Gamma(X_U, \mathscr{F}_{\lambda,\underline{\chi}})_{U'} = R\Gamma\left(\prod_{v\in R} k(v)^\times, R\Gamma(X_{U''}, \mathscr{F}_\lambda)_{U'}\left(\prod_{v\in R}\chi_v\right)\right).$$

This is because $\underline{\chi}$ is trivial on U'' by definition, being a character of $\prod_{v\in R} \mathrm{Iw}_v / \mathrm{Iw}_{v,1}$, so we can drop it from $R\Gamma(X_{U''}, \mathscr{F}_{\lambda,\underline{\chi}})$ and put in the corresponding twist of the U'-action afterwards. Then the above identity comes from the Leray/Hochschild spectral sequence, where by the outer $R\Gamma$ we mean the derived functor of $\prod_{v\in R} k(v)^\times$-invariants. This is useful because we started out with information about Galois representations without the $\underline{\chi}$-twists.

We conclude that the Hecke algebra acting on $R\Gamma(X_{U''}, \mathscr{F}_\lambda)_{U'}$ surjects onto the one acting on $R\Gamma(X_U, \mathscr{F}_{\lambda,\underline{\chi}})_{U'}$. That is, we have a surjection

$$\mathbb{T}^S_{Q,T,R^-}(U'/U'',\lambda) \twoheadrightarrow \mathbb{T}^S_{Q,T}(U'/U,\lambda,\underline{\chi})$$

where by R^- we mean the Hecke algebra includes the action of $(k(v)^\times)^n$ for $v \in R$. (The value of this is that if we have Galois representations associated to things in the source then we also get them in the target.) Then we also have a surjection

$$\mathbb{T}^S_{Q,T,R^-}(U'/U'',\lambda) \twoheadrightarrow \mathbb{T}^S_{Q,T,R^-}(U'',\lambda)$$

with kernel nilpotent of bounded index. So when we state theorems that are usually stated for $\mathbb{T}^S_{Q,T,R^-}(U'',\lambda)$, we'll also be able to deduce them for $\mathbb{T}^S_{Q,T}(U'/U,\lambda,\underline{\chi})$.

9.3 Scholze's theorem restated

Theorem 9.3.1 (Scholze)**.** *1. Suppose $\mathfrak{m} \trianglelefteq \mathbb{T}^S_{Q,T}(U'/U, \lambda, \underline{\chi})$ is a maximal ideal. Then there is a unique continuous semisimple*

$$\overline{r}_\mathfrak{m} : G_F \to GL_n\left(\mathbb{T}^S_{Q,T}(U'/U, \lambda, \underline{\chi})/\mathfrak{m}\right)$$

such that $\overline{r}_\mathfrak{m}$ is unramified outside $S \cup \{w|l\}$ and such that if $v \notin S \cup \{w|l\}$, then $\overline{r}_m(\mathrm{Frob}_v)$ has characteristic polynomial $P_v(x)$, our previously defined element of

$$\mathbb{Z}[q_v^{-1}][GL_n(\mathcal{O}_{F,v})\backslash GL_n(F_v)/GL_n(\mathcal{O}_{F,v})][X].$$

As before, we call $\mathfrak{m}$ non-Eisenstein if $\overline{r}_\mathfrak{m}$ is absolutely irreducible.

2. If $\mathfrak{m}$ is a non-Eisenstein maximal ideal in $\mathbb{T}^S_{Q,T}(U'/U, \lambda, \underline{\chi})$, then there is $N = N([F : \mathbb{Q}], n)$ and $I \trianglelefteq \mathbb{T}^S_{Q,T}(U'/U, \lambda, \underline{\chi})_\mathfrak{m}$ with $I^N = (0)$ and a unique continuous representation

$$r_\mathfrak{m} : G_F \to GL_n\left(\mathbb{T}^S_{Q,T}(U'/U, \lambda, \underline{\chi})_\mathfrak{m}/I\right)$$

which is unramified outside $S \cup \{w|l\}$ and such that if $v \notin S \cup \{w|l\}$, then $r_m(\mathrm{Frob}_v)$ has characteristic polynomial $P_v(X)$.

This theorem explains what the constructed representations look like at good primes. Now we want to write down what they look like at certain bad primes.

Theorem 9.3.2 ([1], slight extension of Scholze)**.** *If either ($v \in R$ and $\sigma \in I_{F_v}$) or ($v \in Q$ and $\sigma \in W_{F_v}$), and if σ has image in $W^{ab}_{F_v} \cong F_v^\times$ equal to $\mathrm{art}_{F_v}(\alpha)$, then $r_\mathfrak{m}(\sigma)$ has characteristic polynomial $P_{v,\alpha}(X)$. (Recall that if $v \in Q$, then we previously defined an element*

$$P_{v,\alpha}(X) \in \mathbb{Z}[q_v^{-1}][\Xi_v]^{S_n}[X]$$

and if $v \in R$ and $\alpha \in \mathcal{O}^\times_{F,v}$, we previously defined

$$P_{v,\alpha}(X) = \prod_{i=1}^n (X - \chi_{v,i}(\alpha)^{-1}) \in \mathcal{O}[X].)$$

The hardest part of proving this is to formulate it; once you do, Scholze's argument works. We might come back to how to prove these things later. Now we know what happens at primes in Q and R, and we want to know what happens at primes that divide the residue characteristic l.

Theorem 9.3.3 ([1])**.** *Keep the above assumptions. Suppose $v|l$ and*

- *l is non-ramified in F.*
- *$l > n^2$.*
- *$v \notin S$.*

- *for all τ, $\lambda_{\tau,1} - \lambda_{\tau,n} + \lambda_{\tau c,1} - \lambda_{\tau c,n} \leq l - 2n - 1$. (Because we're interested in representations that are mod l, mod l^2, etc., not just characteristic 0, we can't talk about representations being crystalline etc. There are two situations in which one has integral versions of local-global compability at primes dividing the residue characteristic: the Fontaine-Laffaile situation and the ordinary situation. We are addressing the Fontaine-Laffaile case first, which is why we require l to be unramified, and this stronger version of the condition that the "spread" of the Hodge-Tate numbers is small compared to l.)*

- *there is $v'|l$, equal to neither v nor $c(v)$, such that*

$$[F_v^+ : \mathbb{Q}_l] + [F_{v'}^+ : \mathbb{Q}_l] < \frac{1}{2}[F^+ : \mathbb{Q}].$$

- *there is a prime $p \neq l$ which splits completely in F, such that no prime above p lies in S and if $\alpha_1, \ldots, \alpha_n$ denote the eigenvalues of $\overline{r}_{\mathfrak{m}}(\mathrm{Frob}_v)$, then $\alpha_i/\alpha_j \neq p$ for all $i \neq j$. (This is called the "(weakly) decomposed generic" condition.)*

- *either $H^\bullet(X_U, \mathscr{F}_{\lambda,\underline{\chi}})_{\mathfrak{m}}[1/l] \neq (0)$, or for all $\tau : F_v \hookrightarrow L$, $\lambda_{\tau,1}+\lambda_{\tau c,1} \leq 0$ and $\lambda_{\tau,n}+\lambda_{\tau c,n} \geq 2n + 2 - l$.*

Then we may suppose (for an appropriate choice of nilpotent ideal I in the previous theorem) that $\overline{r}_{\mathfrak{m}}|_{G_{F_v}}$ is Fontaine-Laffaile, i.e. in the image of the Fontaine-Laffaille functor $\mathbb{G}$, and

$$FL_\tau(\mathbb{G}^{-1}(\overline{r}_{\mathfrak{m}}|_{G_{F_v}})) = \{\lambda_{\tau,1} + n - 1, \ldots, \lambda_{\tau,n}\}.$$

The proof of this depends crucially on a result of Caraiani and Scholze [7] about vanishing of cohomology of unitary groups. In their original paper, in addition to the weakly decomposed generic condition above, they asked for the α_is to also be distinct; they called that condition just decomposed generic. The final version of their paper weakened that condition, and this carries over. This is important in practice.

(Once you linearize the FL module you should be able to determine the characteristic polynomial in the same way, but the big paper didn't write this down.)

9.4 Summary of Fontaine-Laffaile theory

Definition 9.4.1. Fix $a \in \mathbb{Z}^{\mathrm{Hom}_{\mathbb{Q}_l}(F_v,L)}$. An object of the FL category is

- a finitely generated $\mathcal{O}_{F,v} \otimes_{\mathbb{Z}_l} \mathcal{O} \cong \mathcal{O}^{\mathrm{Hom}(F_v,L)}$-module M (note that $M = \bigoplus_\tau M_\tau$ where $M_\tau = M \otimes_{\mathcal{O}_{F,v}\otimes\mathcal{O},\tau\otimes 1} \mathcal{O}$) together with

- a decreasing filtration $\mathrm{Fil}^i M$ by $\mathcal{O}_{F,v} \otimes_{\mathbb{Z}_l} \mathcal{O}$-submodules such that

  - $\mathrm{Fil}^{a_\tau} M = M$ and
  - $\mathrm{Fil}^{a_\tau+l-1} M = (0)$, admitting

- $\mathrm{Frob}_p^{-1} \otimes 1$-linear maps $\Phi^i : \mathrm{Fil}^i M \to M$ (note that $\mathrm{Fil}^i M = \bigoplus_\tau \mathrm{Fil}^i M_\tau$ and we have $\Phi^i : \mathrm{Fil}^i M_\tau \to M_{\tau\circ\mathrm{Frob}_p}$) with

- $\Phi^i|_{\mathrm{Fil}^{i+1} M} = l\Phi^{i+1}$
- $\sum_i \Phi^i \operatorname{Fil}^i M = M$.

We write $\mathscr{MF}^{\underline{a}}$ for the category of these (the maps are what you expect—$\mathcal{O}_{F,v} \otimes_{\mathbb{Z}_l} \mathcal{O}$-linear maps that take Fil^i to Fil^i and commute with the Frobeniuses).

Then there is a functor

$$\mathbb{G} : \mathscr{MF}^{\underline{a}} \to \{\text{finitely generated } \mathcal{O}\text{-modules with a continuous action of } G_{F_v}\}$$

which is fully faithful, exact, has essential image closed under subobjects and quotients (so the point is we're just picking out certain Galois representations that are closed under subobjects and quotients), and satisfies

$$[F_v : \mathbb{Q}_l]\mathrm{length}_{\mathcal{O}}\mathbb{G}(M) = \mathrm{length}_{\mathcal{O}}(M).$$

If M is an object of $\mathscr{MF}^{\underline{a}}$ for two different $\underline{a}$s then the two definitions of $\mathbb{G}(M)$ agree. We have $\mathbb{G}(M \otimes N) \cong \mathbb{G}(M) \otimes \mathbb{G}(N)$ whenever this makes sense.

Definition 9.4.2. If $M \in \mathscr{MF}^{\underline{a}}$, we write $FL_\tau(M)$ for the multiset of integers containing i with multiplicity $\dim_{\mathbb{F}}(\mathrm{gr}^i M_\tau \otimes_{\mathcal{O}} \mathbb{F})$.

If M is l-torsion free, so that $\mathbb{G}(M)$ is a free $\mathcal{O}$-module, then $\mathbb{G}(M)[1/l]$ is crystalline with $HT_\tau(\mathbb{G}(M)[1/l]) = FL_\tau(M)$.

Conversely, if V is a crystalline representation of G_{F_v} over L with $HT_\tau(V) \subset [a_\tau, a_\tau+l-2]$ for all τ, and if $\Lambda \subset V$ is an invariant lattice, then $\Lambda = \mathbb{G}(M)$ for some $M \in \mathscr{MF}^{\underline{a}}$, and M is l-torsion free with the expected FL numbers (equal to $HT_\tau(V)$).

Fontaine-Laffaile theory is a concrete way of working integrally with Galois representations which are "morally" crystalline.

Next time, we will discuss what happens above l in the "ordinary" case. Then we will discuss how to apply this construction of Galois representations to show the concentration of cohomology in few degrees. Then we'll go on to automorphy lifting.

10 April 29: the ordinary case and concentration of cohomology.

10.1 Recap

Let F_0 be an imaginary quadratic field, F^+ a totally real field, and $F = F_0F^+$ a CM field. (For the record, Scholze's theorem only requires that F is imaginary CM. But the theorem about Fontaine-Laffaileness at places dividing l does require F to contain a F_0.) Let $G = \mathrm{res}_{\mathbb{Z}}^{\mathcal{O}_F} GL_n$, X a symmetric space for G (of dimension $[F^+ : \mathbb{Q}]n^2 - 1$), l a prime that splits in F_0 (this is also not needed for Scholze's theorem), $L/\mathbb{Q}_l$ large enough, $\mathcal{O}/\lambda = \mathbb{F}$, $\lambda \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(F,L)}$, V_λ the corresponding algebraic representation of G over $\mathcal{O}$, $U = U_S \times \prod_{v \notin S} GL_n(\mathcal{O}_{F,v})$ an open compact subgroup of $G(\mathbb{A}^\infty) \cong GL_n(\mathbb{A}_F^\infty)$, U' of the same form with $U \trianglelefteq U'$, where S is a finite set of places with $S^c = S$.

CORRECTION: the condition on S we stated before should say that if p lies below an element of S *or* p is ramified in F, p is not split in F_0, and $v|p$, then $v \in S$. "If p is not split in F_0, and one prime above p is considered bad, then all primes above p should be considered bad."

Recall that $S \supset Q \coprod R \coprod T$ and if

$$U_S = \prod_{v \in Q} U_v \times \prod_{v \in R} U_v \times \prod_{v \in T} U_v \times U_{S-Q \cup R \cup T}$$

and similarly for U'_S, then for $v \in Q$, $U'_v = \mathrm{Iw}_v$, $U_v \supset \mathrm{Iw}_{v,1}$; for $v \in R$, $U'_v = U_v = \mathrm{Iw}_v$, and we chose a character $\chi_v : \mathrm{Iw}_v \,/\, \mathrm{Iw}_{v,1} \to \mathcal{O}^\times$; and for $v \in T$ we had some other thing we won't write down again. Places in $Q \cup R$ don't divide l, places in T do. $\rho = \rho_\lambda \otimes \bigotimes_{v \in R} \chi_v$ and $\chi = \bigotimes_{v \in R} \chi_v$. Last time, we were looking at

$$R\Gamma(X_U, \mathscr{F}_\rho = \mathscr{F}_{\lambda, \underline{\chi}})_{U'} \in D^b(\mathcal{O}[U'/U])$$

which has an action of $\mathbb{T}^S_{Q \cup T}(U'/U, \lambda, \underline{\chi})$, a commutative $\mathcal{O}[U'/U]$-algebra and finitely generated $\mathcal{O}$-module. For $\mathfrak{m} \trianglelefteq \mathbb{T}^S_{Q \cup T}(U'/U, \lambda, \underline{\chi})$ a maximal ideal, we (Scholze) constructed

$$\overline{r}_\mathfrak{m} : G_F \to GL_n(\mathbb{T}^S_{Q \cup T}(U'/U, \lambda, \underline{\chi})/\mathfrak{m})$$

and, for $\mathfrak{m}$ non-Eisenstein,

$$r_\mathfrak{m} : G_F \to GL_n(\mathbb{T}^S_{Q \cup T}(U'/U, \lambda, \underline{\chi})_\mathfrak{m}/I)$$

where $I^N = (0)$ for some $N = N([F : \mathbb{Q}], n)$. We controlled the behavior of $r_\mathfrak{m}$ at $v \notin S$, $v \nmid l$, and $v \in Q \cup R$. These are true if F is any CM field; it does not need to contain an F_0 (we need to require that $S^c = S$, and that if v is ramified over $\mathbb{Q}$ and v is not split over F^+ then $v \in S$).

For $v|l$, we need a new method, as provided by Caraiani and Scholze, which gives a new construction of these representations. If $v \notin S$ and λ is "small" compared to l, we saw that $r_\mathfrak{m}|_{G_{F_v}}$ was Fontaine-Laffaile and gave information about the Fontaine-Laffaile numbers. This part really does require $F_0 \subset F$ (though it's "probably just for simplicity in trace formula calculations").

10.2 The ordinary case

Theorem 10.2.1. *Keep the previous assumptions and further suppose*

- $T = \{v|l\}$.
- $\overline{r}_\mathfrak{m}$ *is "decomposed generic" as defined before.*
- $\mathfrak{m} \trianglelefteq \mathbb{T}^S_{R,T}(U'/U, \lambda, \underline{\chi})$ *is ordinary, i.e. for all* $v|l$*, the Hecke operator*

$$U_v^+ = [U_v \operatorname{diag}(\pi_v^{n-1}, \ldots, \pi_v, 1) U_v]$$

is not in $\mathfrak{m}$*. (Though* U_v^+ *depends on the preferred uniformizer* π_v*, the "ordinary" condition does not, because two different* U_v^+*s differ by a unit which corresponds to a diamond operator which has an inverse.)*

(Note that there are many fewer conditions here than for the Fontaine-Laffaile case.)

Then for all $v|l$ and $i = 1, \ldots, n$, there is a unique continuous character

$$\chi_{v,i} : G_{F_v} \to \mathbb{T}^S_{R,T}(U'/U, \lambda, \underline{\chi})^\times_{\mathfrak{m}}$$

such that if $\alpha \in \mathcal{O}^\times_{F,v}$,

$$\chi_{v,i}(\operatorname{art}(\alpha)) = (N_{F_v/\mathbb{Q}_l}\alpha)^{i-1} \prod_{\tau: F_v \hookrightarrow L \text{ continuous}} t(\alpha)^{-\lambda_{\tau,n+1-i}} \langle (1, \ldots, 1, \alpha, 1, \ldots, 1) \rangle$$

where the α on the right is in the ith place, and for the preferred uniformizer π_v,

$$\chi_{v,i}(\operatorname{art}(\pi_v)) = \frac{(N_{F_v/\mathbb{Q}_l}\pi_v)^{i-1}}{l^{[k(v):\mathbb{F}_l](i-1)}} U_{v,i} U^{-1}_{v,i-1}$$

where $U_{v,i}$ is the operator coming from $\operatorname{diag}(\pi_v, \ldots, \pi_v, 1, \ldots, 1)$ *(with i π_vs), and*

1. *for all $\sigma \in G_{F_v}$,*

$$\operatorname{char}_{r_{\mathfrak{m}}(\sigma)}(X) = \prod_{i=1}^n (X - \chi_{v,i}(\sigma))$$

(this is "morally" but imprecisely like saying that $r_{\mathfrak{m}}|^{ss}_{G_{F_v}} = \chi_{v,1} \oplus \cdots \oplus \chi_{v,n}$, but it doesn't actually make sense to semisimplify a representation over a complete local noetherian ring). Furthermore, we should know "what order" the characters are in, whatever that means; what we can actually say is that

2. *for all $\sigma_1, \ldots, \sigma_n \in G_{F_v}$,*

$$(r_{\mathfrak{m}}(\sigma_1) - \chi_{v,1}(\sigma_1))(r_{\mathfrak{m}}(\sigma_2) - \chi_{v,2}(\sigma_2)) \cdots (r_{\mathfrak{m}}(\sigma_n) - \chi_{v,n}(\sigma_n)) = 0$$

in this order—*you can't permute the factors because they don't commute, unless you know that $r_{\mathfrak{m}}(\sigma)$ is upper triangular with diagonal entries $\chi_{v,1}(\sigma), \ldots, \chi_{v,n}(\sigma)$.*

10.3 Which degrees have cohomology

The real dimension of X_U is $[F^+ : \mathbb{Q}]n^2 - 1$. But "most" of its cohomology is clustered in a "small" range around the middle degree. (It might be the case that all the non-Eisenstein cohomology is clustered in this way, but we really don't know.)

Lemma 10.3.1. *Suppose π is an algebraic, generic (don't worry about it), essentially unitary (unitary up to twist by a character; generic plus essentially unitary is true for π_∞ for any cuspidal automorphic representation π of $GL_n(F)$) representation of $GL_n(\mathbb{C})$. (As in previous courses, by this we really mean a $(\mathfrak{g}, K)$-module over* $\operatorname{Lie}(GL_n(\mathbb{C}))$ *etc.) Then*

$$\pi = n - \operatorname{Ind}^{GL_n(\mathbb{C})}_{B_n(\mathbb{C})}(\chi_1, \ldots, \chi_n)$$

where the χ_i are of the form

$$\chi_i : \mathbb{C}^\times \to \mathbb{C}^\times$$
$$z \mapsto z^{m_i} \overline{z}^{w-m_i}$$

for some $m_i, w \in \mathbb{Z}$.

Proof. There is a list of (essentially) unitary irreducible representations of $GL_n(\mathbb{C})$ (maybe due to Vogan; see Theorem 10 of Clozel's Bourbaki talk "recent progress on the classification of the unitary dual of real reductive groups" [8]). (A similar list can be found for $GL_n(\mathbb{R})$, and a similar lemma can be stated.) It's quite long, but you go through it and find that every item except the one in the lemma either fails to be algebraic or fails to be generic.

For example, algebraicity rules out the "complementary series", which are also induced from characters, but ones that can be modified by small powers of $|z|$—the absolute values of the $z^{m_i}\overline{z}^{w-m_i}$ are all $|z|^w$, which doesn't depend on i; in the complementary series you can modify w by a fraction less than 1 that changes for different i, which is not algebraic. Actually, it's believed that the lemma is true without the algebraicity assumption if π arises from a cuspidal automorphic representation.

In some other representations, w can depend on i by an integer quantity, but those representations are "small" and not generic. Specifically, genericity rules out Speh representations, like a character composed with the determinant.

Note that we really mean that π equals the stated induction. It's not just a subquotient. □

Corollary 10.3.2. *If π is as in the lemma, let $GL_n(\mathbb{C})^1 = \{g \in GL_n(\mathbb{C}) \mid |\det(g)| = 1\}$. Then we can describe*

$$H^i\left(\mathrm{Lie}\, GL_n(\mathbb{C})^1, U(n), (\pi(|\det|^2)^{(n-1)/2} \otimes \rho_\lambda)^{\mathbb{R}^\times_{>0}}\right)$$

(we're writing $(|\det|^2)^{(n-1)/2}$ because really the natural normalization of $|\cdot|$ here is the square of the one from high school but that would be confusing to write; by taking $\mathbb{R}^\times_{>0}$-invariants, since it acts on both representations by a character, we just mean that if the central characters of the two are different then we take 0 and if they're the same then we take everything) as follows. It is (0) unless, writing $\lambda \in (\mathbb{Z}^n_+)^{\mathrm{Hom}_{\mathbb{R}}(\mathbb{C},\mathbb{C})} = (\mathbb{Z}^n_+)_1 \times (\mathbb{Z}^n_+)_c$, we have

$$\{m_1, \ldots, m_n\} = \{-\lambda_{1,n}, \ldots, -\lambda_{1,1} - (n-1)\} = \{w + \lambda_{c,1} + n - 1, \ldots, w + \lambda_{c,n}\}$$

and $i \in \left[\frac{n(n-1)}{2}, \frac{n(n-1)}{2} + n - 1\right]$, in which case the cohomology has dimension

$$\binom{n-1}{i - \frac{n(n-1)}{2}}$$

(so the dimensions are all the binomial coefficients for $n-1$ spread over the given range).

Proof. Theorem III.3.3 of Borel-Wallach [4] gives a formula for this cohomology when π is parabolically induced from a character. □

Recall that we decomposed

$$H^i(X_U, \mathscr{F}_\lambda) = \bigoplus_\Phi H^i(X_U, \mathscr{F}_\lambda)_\Phi$$

where Φ ranges over cuspidal types.

Corollary 10.3.3. *Suppose $\Phi = [(G, \pi \| \det \|^{(n-1)/2})]$ is a cuspidal type for G. Then*

$$H^i(X_U, \mathscr{F}_\lambda)_\Phi = (0)$$

unless

1. $\lambda_{\tau,i} + \lambda_{\tau c, n+1-i} = w$ *independent of* τ *and* i,
2. $HC_\tau(\pi) = \{-\lambda_{\tau,n}, \dots, -\lambda_{\tau,1} - (n-1)\}$, *and*
3. $i \in \left[[F^+ : \mathbb{Q}] \frac{n(n-1)}{2}, [F^+ : \mathbb{Q}] \frac{n(n+1)}{2} - 1 \right]$,

in which case it has dimension

$$\dim(\pi^\infty)^U \cdot \binom{[F^+ : \mathbb{Q}]n - 1}{i - [F^+ : \mathbb{Q}] \frac{n(n-1)}{2}}.$$

Where are these numbers coming from? $[F^+ : \mathbb{Q}]n - 1$ is the dimension of the maximal $\mathbb{R}$-split torus in G, modded out by the 1-dimensional maximal $\mathbb{Q}$-split A_G.

Proof. Use the Künneth formula:

$$H^i(\mathfrak{g}_1 \oplus \mathfrak{g}_2, K_1 \times K_2, V_1 \otimes V_2) \cong \bigoplus_{i_1 + i_2 = i} H^{i_1}(\mathfrak{g}_1, K_1, v_1) \otimes H^{i_2}(\mathfrak{g}_2, K_2, V_2)$$

(this is probably also in Borel-Wallach). We have

$$\operatorname{Lie} M_G = \operatorname{Lie} \left(\ker(\operatorname{res}_{\mathbb{Q}}^{F^+} \mathbb{G}_m \xrightarrow{N} \mathbb{G}_m) \right) \oplus \bigoplus_{1}^{[F^+:\mathbb{Q}]} \operatorname{Lie} GL_N(\mathbb{C})^1$$

(because by definition M_G is the set of elements such that the square of the norm of their determinant is 1; since the norm is positive this just means that the norm is 1; then on the left we have the part of the center of G which is split over $\mathbb{R}$, and on the right the part which has compact center over $\mathbb{R}$.) We want to compute

$$H^i \left(\operatorname{Lie} M_g, U(n)^{[F^+:\mathbb{Q}]}, \left(\bigotimes_{i=1}^{[F^+:\mathbb{Q}]} \pi_{v_i} \otimes \rho_{\lambda_{v_i}} \right)^{\mathbb{R}_{>0}^\times} \right)$$

(here $\bigotimes_{i=1}^{[F^+:\mathbb{Q}]}$ ranges over the infinite places of F), and by the Künneth formula this becomes

$$\bigoplus_{i_0 + \dots + i_{[F^+:\mathbb{Q}]} = i} H^{i_0} \left(\operatorname{Lie} \ker(\operatorname{res}_{\mathbb{Q}}^{F^+} \mathbb{G}_m \to \mathbb{G}_m), \{1\}, (\chi_\pi \otimes \chi_{\rho_\lambda})^{\mathbb{R}_{>0}^\times} \right)$$
$$\otimes \bigotimes_{j=1}^{[F^+:\mathbb{Q}]} H^{i_j}(\operatorname{Lie} GL_n(\mathbb{C})^1, U(n), \pi_{v_j} \otimes \rho_{\lambda_{v_j}}).$$

By definition

$$H^i((\mathrm{Lie}\,\mathbb{R}_{>0}^{\times})^e, \mathbb{C}(\chi)) = \begin{cases} 0 & \chi \neq 1 \\ \mathrm{Hom}(\wedge^i \mathbb{R}^e, \mathbb{C}) & \chi = 1 \end{cases}$$

(because the boundary maps in the defining complex are 0). So the previous expression can be rewritten as

$$\bigoplus \mathrm{Hom}(\wedge^i \mathbb{R}^{[F^+:\mathbb{Q}]-1}, \mathbb{C}) \otimes \bigotimes_j H^{i_j}\left(\mathrm{Lie}\,GL_n(\mathbb{C})^1, U(n), (\pi_{v_j} \otimes \rho_{\lambda_j})^{\mathbb{R}_{>0}^{\times}}\right)$$

and we just computed the thing being tensored. We conclude that the dimension is either 0 or

$$\sum_{i_0+\cdots+i_{[F^+:\mathbb{Q}]}=i} \binom{[F^+:\mathbb{Q}]-1}{i_0}\binom{n-1}{i_1 - \frac{n(n-1)}{2}} \cdots \binom{n-1}{i_{[F^+:\mathbb{Q}]} - \frac{n(n-1)}{2}} = \binom{[F^+:\mathbb{Q}]n-1}{i - [F^+:\mathbb{Q}]\frac{n(n-1)}{2}}.$$

(Maybe we didn't actually have to split everything up by infinite places. The point is that the cohomology has to vanish unless π is fully induced from algebraic characters on a Borel, and then it can be explicitly calculated.) □

Corollary 10.3.4. *If π is a regular algebraic cuspidal automorphic rep of $GL_n(\mathbb{A}_F)$ (where F is CM), and $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$, then there is a unique continuous semisimple representation $r_l(\pi) : G_F \to GL_n(\overline{\mathbb{Q}}_l)$ such that for almost all v, $r_l(\pi)(\mathrm{Frob}_v)$ and $\mathrm{rec}(\pi_v)(\mathrm{Frob}_v)$ have the same characteristic polynomial.*

(Actually this was already proven before Scholze's theorem, but it also follows from it.)

Proof. $\pi\|\det\|^{(n-1)/2}$ contributes to the cohomology of X_U for some U. □

Proposition 10.3.5. *Suppose $\mathfrak{m}$ is a non-Eisenstein maximal ideal of $\mathbb{T}^S_{R,T}(U, \lambda, \underline{\chi})$. Then*

$$H^i(R\Gamma(X_U, \mathscr{F}_{\lambda,\underline{\chi}}))_{\mathfrak{m}}[1/l] = (0)$$

unless $\lambda_{\tau c,i} + \lambda_{\tau,n+1-i} = w$ is independent of τ and i, and $i \in [q_0, q_0 + l_0]$ where

- *$q_0 = [F^+ : \mathbb{Q}]n(n-1)/2$ and*
- *$l_0 = n[F^+ : \mathbb{Q}] - 1$.*

Proof. Calculate this using cuspidal types. We just saw that the proposition is true for $H^i(R\Gamma(X_U, \mathscr{F}_{\lambda,\underline{\chi}}))_{\mathfrak{m}}[1/l]_\Phi$ if $\Phi = [(G, \pi)]$. We just need $H^i(R\Gamma(X_U, \mathscr{F}_{\lambda,\underline{\chi}}))_{\mathfrak{m}}[1/l]_\Phi$ for $\Phi = [(L, \pi)]$ where $L \neq G$. Suppose

$$L = \mathrm{res}^F_{\mathbb{Q}} GL_{n_1} \times \cdots \times \mathrm{res}^F_{\mathbb{Q}} GL_{n_r}$$

for $n = n_1 + \cdots + n_r$, $r > 1$, $\pi = \pi_1 \otimes \cdots \otimes \pi_r$. Since $\pi\|\det\|^{(1-n)/2}$ is regular algebraic, $\pi_i\|\det\|^{(1-n)/2}$ is regular algebraic for all i. Also

$$\mathrm{rec}(\pi_v|\cdot|_v^{(1-n)/2}) = \bigoplus \mathrm{rec}(\pi_{i,v}|\cdot|_v^{(1-n)/2})$$

so

$$r_l(\pi\|\cdot\|^{(1-n)/2})^{ss} = \bigoplus r_l(\pi_i\|\cdot\|^{(1-n)/2})^{ss}$$

because by the theorem applied to GL_{n_i}, the two sides have the same characteristic polynomial on Frobenius elements, which are dense. So $\overline{r}_{\mathfrak{m}}$, the reduction of $r_l(\pi\|\cdot\|^{(1-n)/2})^{ss}$, is reducible. This contradicts $\mathfrak{m}$ being non-Eisenstein. That is, proper parabolic subgroups can't contribute to the cohomology here. $\square$

It's natural to conjecture that the proposition is true without inverting l—that for non-Eisenstein ideals, even torsion concentrates in few degrees—but we don't know how to do that.

11 May 4: starting automorphy lifting.

We will state the main theorem we're going to prove, which is long and complicated, followed by a cleaner corollary. Then we will spend the rest of today showing how the corollary follows from the theorem.

11.1 Statement of theorem

Suppose that F_0 is an imaginary quadratic field, F^+ a totally real field not $\mathbb{Q}$, and $F = F^+F_0$. Here is a condition that Richard thinks is unnecessary, but he hasn't checked everything about removing it yet: if p is ramified in F, then p splits in some imaginary quadratic subfield (so F actually contains more than one imaginary quadratic subfield, because if it had a unique one, a prime that ramifies in F_0 would ramify in F but not split in F_0).

Let $n \geq 2$. Let l be a prime such that l splits in F_0, $l > n^2$, l is non-ramified in F (this is all for the Fontaine-Laffaile case), and for all $v|l$, there is $v'|l$, $v' \neq v, v^c$ such that

$$[F_v^+ : \mathbb{Q}_l] + [F_{v'}^+ : \mathbb{Q}_l] < \frac{1}{2}[F^+ : \mathbb{Q}_l]$$

(for example if no prime has a place over it of more than 1/4 of the degree of F). Fix $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$.

Let R be a finite set of places of F such that $R^c = R$, $v \in R$ implies $q_v \equiv 1 \pmod{l}$, and $v \in R$ and $v|p$ implies p is split in F_0. Let $\underline{\lambda} \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(F,\overline{\mathbb{Q}}_l)}$ be such that

$$\lambda_{\tau,1} - \lambda_{\tau,n} + \lambda_{\tau c,1} - \lambda_{\tau c,n} < l - 2n$$

for all τ. Let $r : G_F \to GL_n(\overline{\mathbb{Q}}_l)$ be a continuous representation such that

- r is unramified outside $R \cup \{v|l\}$,
- if $v|l$, r is crystalline with HT numbers $\{\lambda_{\tau,1} + n - 1, \ldots, \lambda_{\tau,n-1} + 1, \lambda_{\tau n}\}$,
- if $v \in R$, then $r|_{I_{F_v}}$ is unipotent and $\overline{r}|_{G_{F_v}}$ is trivial ("r has the simplest ramification possible"). This can be achieved by finite base change: any finite group in the image of inertia can be removed by base change, as can the image of $\overline{r}$.

Further assume:

a. Let $\tilde{F}$ denote the normal closure of F over $\mathbb{Q}$. Then there exists $\sigma \in G_{\tilde{F}}$ such that $(\mathrm{ad}\,\overline{r})^\tau(1)^{\langle\sigma\rangle} = (0)$ for all $\tau \in \mathrm{Gal}(\tilde{F}/\mathbb{Q})$, where by $(\mathrm{ad}\,\overline{r})^\tau$ we mean $(\mathrm{ad}\,\overline{r}) \circ \mathrm{conj}_\tau$, by (1) we mean twist by the cyclotomic character, and by $\langle\sigma\rangle$ we mean we take the σ-fixed points.

   This implies by Chebotarev that there is $p \neq l$ completely split in F such that r is unramified above p and for all $v|p$, $(\mathrm{ad}\,\overline{r})(1)^{\mathrm{Frob}_v} = (0)$. Then if $\overline{r}(\mathrm{Frob}_v)$ has eigenvalues $\alpha_1, \ldots, \alpha_n$, we have $(\alpha_i/\alpha_j)p^{-1} \neq 1$ for all i, j, since $(\alpha_i/\alpha_j)p^{-1}$ is an eigenvalue of Frob_v on $(\mathrm{ad}\,\overline{r})(1)$. So we deduce the decomposed generic condition.

   Furthermore, $H^0(G_{F_v}, (\mathrm{ad}\,\overline{r})(1)) = (0)$, which implies by Tate local duality that

$$H^2(G_{F_v}, \mathrm{ad}\,\overline{r}) = (0)$$

   (the two cohomologies pair to the cyclotomic character).

   Decomposed genericity is slightly weaker than the condition we started with, since the prime p we found by Chebotarev above cannot be $\equiv 1 \pmod{l}$, or we will certainly have $(\alpha_i/\alpha_j)p^{-1} \neq 1$ for $i = j$. So decomposed genericity is equivalent to the condition we started with when $p \not\equiv 1 \pmod{l}$, but if you just assume decomposed genericity we could have chosen $p \equiv 1 \pmod{l}$. But if we did that then we'd have to assume the cohomology vanishing separately.

b. $H = \overline{r}(G_{F(\zeta_l)})$ is "enormous", i.e.

   1. H is absolutely irreducible. (Recall that $H \subset GL_n(\overline{\mathbb{F}}_l)$.)
   2. $H^0(H, \mathrm{ad}\,\overline{r}) = \overline{\mathbb{F}}_l$ and $H^1(H, \mathrm{ad}\,\overline{r}) = (0)$.
   3. For all H-submodules $(0) \neq W \subset \mathrm{ad}^0\,\overline{r}$ (the trace 0 part), there is a regular semisimple (diagonalizable with distinct eigenvalues) $h \in H$ with $W^{\langle h\rangle} \neq (0)$.

Finally, suppose that there is a regular algebraic cuspidal automorphic representation π of $GL_n(\mathbb{A}_F)$ such that

- $\overline{r}_l(\pi) \cong \overline{r}$,
- π is unramified outside R,
- for all $v \in R$, $\pi_v^{\mathrm{Iw}_v} \neq (0)$,
- $HC_{\iota\circ\tau}(\pi) = \{-\lambda_{\tau,n}, -\lambda_{\tau,n-1} - 1, \ldots, -\lambda_{\tau,1} + 1 - n\}$ for all $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$.

In summary, there are some conditions on the field F, there are splitting conditions on the prime l, the bad places can only be bad in the simplest possible manner, and the weights can't be too spread-out because of the Fontaine-Laffaile condition. The Galois representation is as unramified as possible. It has to be decomposed generic because we want local-global compatibility at the primes dividing l, which depends on Caraiani-Scholze. It has to be "enormous" because we're going to make a Chebotarev argument and will need those properties to find the special primes we need. Finally, we assume that r is residually automorphic, ocming from a regular algebraic cuspidal automorphic representation which is again as unramified as it reasonably could be.

Theorem 11.1.1. *With these assumptions, r is automorphic.*

Now for the cleaned-up consequence we'll prove from the theorem today, which follows from the theorem plus base change.

Corollary 11.1.2. *Suppose F is any CM field (possibly $F = F^+$), $n \geq 2$, $l > n^2$ is a prime unramified in F, $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$, $\underline{\lambda} \in (\mathbb{Z}^n_+)^{Hom(F,\overline{\mathbb{Q}}_l)}$, and for all τ,*

$$\lambda_{\tau,1} - \lambda_{\tau,n} + \lambda_{\tau c,1} - \lambda_{\tau c,n} < l - 2n.$$

Let $r : G_F \to GL_n(\overline{\mathbb{Q}}_l)$ be unramified almost everywhere and crystalline with HT numbers λ above l. Suppose $\overline{r}$ satisfies (a) and (b) above.

Suppose π is a regular algebraic cuspidal automorphic representation of F such that $\overline{r}_l(\pi) \cong \overline{r}$, π is unramified above l, and $HC_{\iota\tau}(\pi) = \{-\lambda_{\tau,n}, \ldots, -\lambda_{\tau,1} + 1 - n\}$ for all $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$.

Then r is automorphic.

11.2 Theorem implies corollary

The following is a (non-obvious) consequence of base change.

Proposition 11.2.1. *Suppose E/F is a solvable extension of CM fields. Suppose also that $r : G_F \to GL_n(\overline{\mathbb{Q}}_l)$ is a continuous representation with $r|_{G_E}$ irreducible and automorphic. Then r is automorphic.*

So we are going to look for a solvable extension of F such that when we restrict the situation in the corollary to that extension of F, then it satisfies the stronger assumptions of the theorem. Since over that extension r is automorphic, by the proposition it is already automorphic over F. In particular we need to choose the solvable extension in such a way that the conditions (a) and (b) don't change.

(b) remains true under a base change E/F if E is linearly disjoint from $\overline{F}^{\ker \overline{r}}(\zeta_l)$ over F. This is because in this case $\mathrm{Gal}(E\overline{F}^{\ker \overline{r}}(\zeta_l)/E) = \mathrm{Gal}(\overline{F}^{\ker \overline{r}}(\zeta_l)/F)$ and the enormous condition was just on the second Galois group.

(a) remains true under a base change E/F if the normal closure $\tilde{E}$ of E over $\mathbb{Q}$ is linearly disjoint (over $\overline{F}$) from the normal closure $\widetilde{\overline{F}^{\ker \mathrm{ad}\, \overline{r}(1)}}$ of $\overline{F}^{\ker \mathrm{ad}\, \overline{r}(1)}$ over $\mathbb{Q}$. This is for a similar reason: the condition is on the extension $\widetilde{\overline{F}^{\ker \mathrm{ad}\, \overline{r}(1)}}$ over F, and you can multiply everything by $\tilde{E}$.

How do we choose E to satisfy this? For each $\overline{F}^{\ker \overline{r}}(\zeta_l) \supset K \supsetneq F$, choose a place v_K of F which does not split in K. Let V_2 be the (finite) set of such places v_K.

Similarly, for each $\widetilde{\overline{F}^{\ker \mathrm{ad}\, \overline{r}(1)}} \supset K \supsetneq \tilde{F}$, choose a rational prime p_K which splits completely in $\tilde{F}$ but not in K. Let V_1 be the set of places of F above the primes p_K.

If E/F is an extension in which all the places in $V_1 \cup V_2$ split completely, then $E \cap \overline{F}^{\ker \overline{r}(\zeta_l)}$ contains F and is an example of one of the subextensions K we went through to construct V_2, unless it equals F. But all the primes in V_2 split completely in $E \cap \overline{F}^{\ker \overline{r}(\zeta_l)}$, because they split completely in E. We conclude that $E \cap \overline{F}^{\widetilde{\ker \bar{r}(\zeta_l)}} = F$. Similarly, each p_K splits completely in E over $\mathbb{Q}$, hence in $\tilde{E}$ over $\mathbb{Q}$, and $\tilde{E} \cap \overline{F}^{\widetilde{\ker \operatorname{ad} \bar{r}(1)}}$ contains $\tilde{F}$, hence equals $\tilde{F}$. If in addition $\widetilde{E/F}$ is Galois then E is linearly disjoint from $\overline{F}^{\ker \bar{r}(\zeta_l)}$ over F and $\tilde{E}$ is linearly disjoint from $\overline{F}^{\ker \operatorname{ad} \bar{r}(1)}$ over $\tilde{F}$.

Since we are choosing primes with desired splitting behavior using Chebotarev, which gives us infinitely many choices, we may assume that the primes in $V_1 \cup V_2$ stay away from any finite set of primes, in particular those where F or π or r is ramified or those dividing l.

Lemma 11.2.2. *Suppose F is a number field and S a finite set of places of F. For $v \in S$, let E_v/F_v be a finite Galois extension. Then there is a finite solvable Galois extension E/F such that for all $v \in S$ and for all $w|v$ of E, $E_w/F_v \cong E_v/F_v$. That is, there is $E_w \xrightarrow{\sim} E_v$ extending the identity on F_v.*

Proof idea. This comes from class field theory using an induction argument. The base case is when all the extensions are cyclic, hence cut out by some finite order character of $F_v^\times$. You put those characters together into some global character. Then you use the congruence subgroup property for groups of units in number fields: in the finitely generated group $\mathcal{O}_F[1/S]^\times$, any finitely generated subgroup is cut out by a congruence condition. □

Now WLOG F contains an imaginary quadratic field F_0, because we can replace F by $F(\sqrt{-p})$ where $p \neq l$ (since we still need l to be unramified) and the primes below $V_1 \cup V_2$ are all split in $\overline{\mathbb{Q}}(\sqrt{-p})$ (hence in $F(\sqrt{-p})$). We can find such a p by quadratic reciprocity, e.g. by choosing it to be $\equiv -1$ mod the primes below $V_1 \cup V_2$ and whatever mod l. So now we've preserved all the other conditions in the corollary and arranged for F to contain F_0 (and it will still be CM because it's the compositum of two CM fields).

Now we find E^+/F^+ a solvable Galois representation such that

1. the infinite places of F^+ split completely (so E^+ is totally real). (Our previous lemma doesn't say that if F is CM we can choose E to be CM. But if F is totally real we can choose E to be totally real by requiring that $E_v = F_v$ for v at infinity. So we can do that for F^+ and then take the compositum with F.)

2. primes below $V_1 \cup V_2$ split in E^+/F^+.

3. E^+/F^+ is unramified above l and if $v|l$ is a place of E^+ then $[E_v^+ : \mathbb{Q}_l] < \frac{1}{4}[E^+ : \mathbb{Q}]$. (For example, this is satisfied if primes above l split completely and $[E^+ : F^+] \geq 4$.)

4. if $v \nmid l$ then, writing $E = E^+F$, either $\pi_{E,v}$ is unramified and $r|_{G_{E_v}}$ is unramified, or

   - $\pi_{E,v}^{\mathrm{Iw}_v} \neq (0)$. (Theorem: if π is an irreducible smooth representation of $GL_n(K)$ for K a p-adic field, then there is L/K a finite extension such that for all $M \supset L$ finite, $\pi_M = BC_K^M(\pi)$ has an Iwahori fixed vector. This follows from local Langlands if you know that having an Iwahori fixed vector follows from the Galois representation having unipotent ramification.)
   - $r|_{I_{F_v}}$ is unipotent. (The image of the inertia group modulo the maximal unipotent subgroup will always be finite, so a finite base change will remove that ramification and make the ramification unipotent.)

- $q_v \equiv 1 \pmod{l}$ (it suffices to require $\zeta_l \in E_v^+$).
- $\overline{r}|_{G_{E_v}} = 1$ (obtainable by finite base change).
- v is split in E/E^+ (obtainable by finite base change).

  Then R can be the set of the primes from the above list.

5. $[E^+ : \mathbb{Q}] > 1$.

6. if v is ramified in F, then $F_v \subset E_v^+$.

We can find such an E^+ by the lemma above. (We can ensure that $[E^+ : F^+] > X$ by imposing $[E_v^+ : F_v^+] > X$ for some v.)

Replacing F by E, we have ensured all the conditions of the theorem except

- if p is ramified in F, then p is split in some imaginary quadratic subfield of F.

- if p lies below R, then p is split in some imaginary quadratic subfield of F.

- l is split in some imaginary quadratic subfield of F.

Let Ω be these primes together with the rational primes below $V_1 \cup V_2$ (which we also need to keep split). Now replace F by $F(\sqrt{-p_0})$ where $p_0 \equiv 1 \pmod{4}$ and

$$p_0 \equiv -1 \left(\bmod \prod_{\substack{p \in \Omega \\ p \neq 2}} p \right).$$

If $p \in \Omega$, $p \neq 2$ then $\left(\frac{-p_0}{p}\right) = 1$. Then p splits in $\mathbb{Q}(\sqrt{-p_0})$. Then all conditions will be satisfied except

- 2 may not split in an imaginary quadratic subfield.

- p_0 may not split in an imaginary quadratic subfield.

So we're going to try $F(\sqrt{-p_0}, \sqrt{-p_1}, \sqrt{-p_2})$, where we look for p_1, p_2 such that

- all primes below $V_1 \cup V_2$ are split in $\mathbb{Q}(\sqrt{-p_1})$ and $\mathbb{Q}(\sqrt{-p_2})$.

- $p_1, p_2 \neq l$. (For this and the previous condition, it suffices to set

$$p_1, p_2 \equiv -1 \pmod{l \times (\text{primes dividing } V_1 \cup V_2))}.$$

- p_1, p_2 are split in $\mathbb{Q}(\sqrt{-p_0})$. (So we need p_2 to be a quadratic residue mod p_0.)

- p_0 splits in $\mathbb{Q}(\sqrt{-p_1})$. (For p_1 to satisfy this and the previous bullet point, we can set $p_1 \equiv -1 \pmod{p_0}$ and $p_1 \equiv 1 \pmod 4$. This ensures that p_0 splits in $\mathbb{Q}(\sqrt{-p_1})$, and also making $p_1 \equiv 1 \pmod 4$ is compatible with what we've already asked for. We need to check that p_1 splits in $\mathbb{Q}(\sqrt{-p_0})$. But since $p_0, p_1 \equiv 1 \pmod 4$,

$$\left(\frac{-p_0}{p_1}\right) = \left(\frac{p_1}{p_0}\right) = \left(\frac{-p_1}{p_0}\right) = 1$$

once we've set $p_1 \equiv -1 \pmod{p_0}$, as desired.)

- 2 splits in $\mathbb{Q}(\sqrt{-p_2})$. (So $p_2 \equiv -1 \pmod 8$.)

We declare victory. Next time we will start proving the main theorem.

12 May 6: Hecke operator properties and lifting setup.

12.1 A few more properties of Hecke operators

1. Let $\Delta \supset U' \supset U$ where Δ is a semigroup and U', U are neat open compact subgroups. Let $\rho : \Delta \to \mathrm{Aut}(F)$ be a representation, giving rise to the sheaf $\mathscr{F}_\rho$ over X_U or $X_{U'}$. The natural map $X_U \to X_{U'}$ corresponds to the map of cohomology in the other direction

$$i_{U/U'} = [UU'] : R\Gamma(X_{U'}, \mathscr{F}_\rho) \to R\Gamma(X_U, \mathscr{F}_\rho)$$

which coincides with the Hecke operator $[UU']$ because if you decompose UU' into left U'-cosets, you just get the single coset U'. We also have a map

$$\mathrm{tr}_{U/U'} = [U'U] : R\Gamma(X_U, \mathscr{F}_\rho) \to R\Gamma(X_{U'}, \mathscr{F}_\rho)$$

and $\mathrm{tr}_{U/U'} \circ i_{U/U'} = [U' : U]$. Let

$$i : \mathbb{Z}\left[\frac{1}{[U' : U]}\right][U'\backslash\Delta/U'] \to \mathbb{Z}\left[\frac{1}{[U' : U]}\right][U\backslash\Delta/U]$$
$$[U'gU'] \mapsto \frac{1}{[U' : U]}\sum_i [Ug_iU]$$

where we decompose $U'gU' = \coprod_i Ug_iU$. Then you can show that for a Hecke operator T,

$$T \circ \mathrm{tr}_{U/U'} = \mathrm{tr}_{U/U'} \circ i(T)$$
$$i_{U/U'} \circ T = i(T) \circ i_{U/U'}.$$

This allows us to compare Hecke operators on $R\Gamma(X_U, \mathscr{F}_\rho)$ and $R\Gamma(X_{U'}, \mathscr{F}_\rho)$ under $i_{U/U'}$ and $\mathrm{tr}_{U/U'}$.

2. Suppose $U' = (U' \cap gU'g^{-1})U$ and $U \cap gU'g^{-1} = U \cap gUg^{-1}$ (*). This implies that if $UgU = \coprod g_i U$, then $U'gU' = \coprod g_i U'$—the two coset decompositions are the same. Then

$$[UgU] i_{U/U'} = i_{U/U'} [U'gU'].$$

(This is not a special case of the previous statement. If we wanted to apply the previous statement we would write

$$U'gU' = \coprod_{v_i \in U'/U} Ugv_i U.$$

We could plug these in and get some equation. The conditions ensure that each of the double cosets Ugv_iU is the same, and if you plug this into the second equation in the previous statement you would get $[UgU]i_{U/U'} = i_{U/U'}[U'gU']$ but multiplied by $[U' : U]$. The claim is that in this case, we can divide.)

For example, if $U = \mathrm{Iw}_{v,1}$ and $U' = \mathrm{Iw}_v$, and $g \in GL_n(\mathcal{O}_{F,v})$ or $g = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$ for $v(\alpha_1) \geq \cdots \geq v(\alpha_n)$, then the conditions (*) hold. For g diagonal you can argue that the only difference between U' and U is that U' doesn't have conditions on the diagonal entries, but multiplying by $U' \cap gU'g^{-1}$ gives you those degrees of freedom back; similarly the conditions on the diagonal entries are the only difference between $gU'g^{-1}$ and gUg^{-1}, and intersecting with U gives you those conditions back. For $g \in GL_n(\mathcal{O}_{F,v})$ you can assume that g is a permutation matrix (the Bruhat decomposition says that the double cosets of Iw_v in $GL_n(\mathcal{O}_{F,v})$ are all represented by such things) and again compare the matrix entry conditions.

But there are many cases where the conditions (*) don't hold. For example, if $U = \mathrm{Iw}_v$, $U' = GL_n(\mathcal{O}_{F,v})$, $n = 2$, $F_v = \mathbb{Q}_p$, then

$$\mathrm{Iw}_v \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \mathrm{Iw}_v = \coprod_{i \in \mathbb{Z}/p\mathbb{Z}} \begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix} \mathrm{Iw}_v$$

$$GL_2(\mathbb{Z}_p) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} GL_2(\mathbb{Z}_p) = \coprod_{i \in \mathbb{Z}/p\mathbb{Z}} \begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix} GL_2(\mathbb{Z}_p) \coprod \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} GL_2(\mathbb{Z}_p).$$

These are the operators U_p and T_p respectively, which we are used to not being compatible for classical modular forms, so this isn't surprising.

3. The previous statement is for $i_{U/U'} : R\Gamma(X_{U'}, \mathscr{F}_\rho) \to R\Gamma(X_U, \mathscr{F}_\rho)$, but when U is normal in U', we would also like it for the version with $\mathcal{O}[U'/U]$-structure. Suppose $\Delta \supset U' \trianglerighteq U$ are as before. Suppose that for all $g \in \Delta$, we have

   - $U' = U(U' \cap g^{-1}U'g)$
   - $U' \cap gUg^{-1} \subset U$
   - $U \cap gU'g^{-1} = U \cap gUg^{-1}$

   and for all $g \in \Delta$ and $U \in g^{-1}U'g \cap U'$, we have $gug^{-1}u^{-1} \in U$. Then the actions of $[U'gU']$ on $R\Gamma(X_{U'}, \mathscr{F}_\rho)$ and $[UgU]$ on $R\Gamma(X_U, \mathscr{F}_\rho)_{U'}$ in $D^b(\mathcal{O}[U'/U])$ are compatible under the natural map

$$R\Gamma(X_{U'}, \mathscr{F}_\rho) \to R\Gamma(X_U, \mathscr{F}_\rho)_{U'}$$

and more strongly, under the isomorphism

$$R\Gamma(X_{U'}, \mathscr{F}_\rho) \xrightarrow{\sim} R\Gamma(U'/U, R\Gamma(X_U, \mathscr{F}_\rho)_{U'}) \in D^b(\mathcal{O}[U'/U]).$$

For example, the assumptions hold if $U = \mathrm{Iw}_{v,1}$, $U' = \mathrm{Iw}_v$, and $\Delta = U\Xi_v^+U$ (recall $\Xi_v^+ = \{\mathrm{diag}(\alpha_1, \dots, \alpha_n) \mid v(\alpha_1) \geq \cdots \geq v(\alpha_n)\}$) using the same argument as before.

12.2 Bernstein's structure of the Iwahori-Hecke algebra

Assume $l \nmid q_v$ (we need $q_v \in \mathcal{O}^\times$). Then we have an isomorphism

$$\begin{aligned} \mathcal{O}[\mathrm{Iw}_v \backslash GL_n(F_v)/\mathrm{Iw}_v] &\cong \mathcal{O}[\mathbb{Z}^n]\widetilde{\otimes}_{\mathcal{O}}\mathcal{O}[\mathrm{Iw}_v \backslash GL_n(\mathcal{O}_{F,v})/\mathrm{Iw}_v] \\ t_i = t_{v,\pi_v,i} &\mapsto (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}^n \\ q_v^{in-\frac{i(i+1)}{2}}[\mathrm{Iw}_v \,\mathrm{diag}(\pi_v, \dots, \pi_v, 1, \dots, 1)\,\mathrm{Iw}_v] &\mapsto (1, \dots, 1, 0, \dots, 0) \end{aligned}$$

where by $\widetilde{\otimes}$ we mean the module is the usual tensor product but the multiplication is twisted in a way we'll describe later, in the first formula the 1 on the right is in the ith position, and in the second formula there are i π_vs on the left and i 1s on the right. Furthermore, modding $GL_n(\mathcal{O}_{F,v})$ out by an appropriate normal subgroup contained in Iw_v, we can see that

$$\mathcal{O}[\mathrm{Iw}_v \backslash GL_n(\mathcal{O}_{F,v})/\mathrm{Iw}_v] \cong \mathcal{O}[B_n(k(v))\backslash GL_n(k(v))/B_n(k(v))]$$

where B_n is the upper triangular Borel. Recall that $B_n(k(v))\backslash GL_n(k(v))/B_n(k(v))$ has a basis given by the elements $S_w = B_n(k(v))wB_n(k(v))$ where $w \in S_n$ is a permutation matrix (this is the Bruhat decomposition). There's a length function $l : S_n \to \mathbb{Z}_{\geq 0}$ taking w to the length of the shortest expression for w using only the transpositions $(i, i+1)$ for $i = 1, \dots, n-1$. If $w_1, w_2 \in S_n$ and $l(w_1, w_2) = l(w_1) + l(w_2)$, then $S_{w_1}S_{w_2} = S_{w_1w_2}$. Therefore $\mathcal{O}[\mathrm{Iw}_v \backslash GL_n(\mathcal{O}_{F,v})/\mathrm{Iw}_v]$ is generated as an algebra by the $S_{(i,i+1)}$s.

The multiplication structure on the twisted tensor product looks like

$$\begin{aligned} S_{(j,j+1)}t_i &= t_iS_{(j,j+1)} \text{ unless } i = j \text{ or } j+1 \\ S_{(i,i+1)}t_i &= t_{i+1}S_{(i,i+1)} + (q_v - 1)t_i \\ S_{(i,i+1)}t_{i+1} &= t_iS_{(i,i+1)} + (1 - q_v)t_i. \end{aligned}$$

Furthermore

$$\begin{aligned} S_{(i,i+1)}^2 &= (q_v - 1)S_{(i,i+1)} + q_v \\ S_{(i,i+1)}S_{(j,j+1)} &= S_{(j,j+1)}S_{(i,i+1)} \text{ if } i \neq j \pm 1 \\ S_{(i,i+1)}S_{(i+1,i+2)}S_{(i,i+1)} &= S_{(i+1,i+2)}S_{(i,i+1)}S_{(i+1,i+2)} \end{aligned}$$

and these relations generate all the relations. (When $q_v = 1$, these degenerate to the generators and relations giving the symmetric group, and we just get the group algebra of the symmetric group. So this is a "deformation" of that group algebra. If you then also include the t_is, you get the group algebra of $\mathbb{Z}^n \rtimes S_n$ where S_n acts on $\mathbb{Z}^n \rtimes S_n$ by the permutation action.) Furthermore,

$$Z(\mathcal{O}[\mathrm{Iw}_v \backslash GL_n(F_v)/\mathrm{Iw}_v]) = \mathcal{O}[\mathbb{Z}^n]^{S_n}$$

where S_n has the usual permutation action on $\mathbb{Z}^n$ (this is clear when $q_v = 1$ as described before, and Bernstein says it's always true). Set

$$\mathrm{tr} = \sum_{w \in S_n} S_w$$

so for example $S_w \,\mathrm{tr} = q_v^{l(w)} \,\mathrm{tr} = \mathrm{tr}\, S_w$. Now look at our previously defined

$$i : \mathcal{O}[GL_n(\mathcal{O}_{F,v}) \backslash GL_n(F_v)/GL_n(\mathcal{O}_{F,v}) \to \mathcal{O}[\mathrm{Iw}_v \backslash GL_n(F_v)/\mathrm{Iw}_v]$$

and consider the element $T_{v,j}$ of the source corresponding to $\mathrm{diag}(\pi_v, \ldots, \pi_v, 1, \ldots, 1)$, with j π_vs. Then

$$i(q_v^{j(j-1)/2} T_{v,j})[GL_n(\mathcal{O}_{F,v}) : \mathrm{Iw}_v] = S_j(t_1, \ldots, t_n)\,\mathrm{tr}$$

where S_j is the jth elementary symmetric function. In particular, this gives an isomorphism between the unramified Hecke algebra and the center of the Iwahori-Hecke algebra, with the jth generator of the unramified Hecke algebra corresponding to the jth elementary symmetric funciton.

Consider the special case $q_v \equiv 1 \pmod{l}$. Reducing mod λ, we have an isomorphism

$$\begin{aligned}\mathbb{F}[\mathrm{Iw}_v \backslash GL_n(\mathcal{O}_{F,v})/\mathrm{Iw}_v] &\cong \mathbb{F}[S_n] \\ [\mathrm{Iw}_v\, w\, \mathrm{Iw}_v] &\mapsto w\end{aligned}$$

since now $S_{(i,i+1)}^2 = 1$, and

$$\mathbb{F}[\mathrm{Iw}_v \backslash GL_n(F_v)/\mathrm{Iw}_v] \cong \mathbb{F}[\mathbb{Z}^n \rtimes S_n].$$

Remark 2.

$$\begin{aligned}[GL_n(\mathcal{O}_{F,v}) : \mathrm{Iw}_v] &= [GL_n(k(v)) : B_n(k(v))] \\ &= \frac{(q_v^n - 1)\cdots(q_v^n - q_v^{n-1})}{(q_v - 1)^n q_v^{n(n-1)/2}} \\ &= \frac{(q_v^n - 1)(q_v^{n-1} - 1)\cdots(q_v - 1)}{(q_v - 1)^n} \\ &= (1 + \cdots + q_v^{n-1})(1 + \cdots + q_v^{n-2})\cdots(1 + q_v)1 \\ &\equiv n! \pmod{l}.\end{aligned}$$

So if $l > n$, then $[GL_n(\mathcal{O}_{F,v}) : \mathrm{Iw}_v] \in \mathcal{O}^\times$. Then the formula for $T_{v,j}$ we just gave really determines $T_{v,j}$, since you can divide through by $[GL_n(\mathcal{O}_{F,v}) : \mathrm{Iw}_v]$.

12.3 The main theorem again

F_0 is imaginary quadratic, $F^+ \neq \mathbb{Q}$ is totally real, and $F = F^+ F_0$. If p is ramified in F, we required p to split in some imaginary quadratic subfield of F. Despite what we said last time, this actually is needed as things stand. The issue is that to construct Galois representations and check local-global compatibility, we need to base change from $GL_n(F)$ to a $2n$-variable unitary group, then base change again to GL_{2n}, then descend again to another unitary group, and the only complete written proof of these base changes was given by Sug Woo Shin, who imposed this condition. More general theorems have been published, but they depend on Arthur's unwritten work.

We chose l a prime with various conditions, $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$, R a finite set of bad places (with conditions like $q_v \equiv 1 \pmod{l}$), $\underline{\lambda} \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(F,L)}$, $r : G_F \to GL_n(\overline{\mathbb{Q}}_l)$ unramified outside l and R, crystalline above l in the Fontaine-Laffaile range, etc., such that for $v \in R$, $r|_{I_{F_v}}$ is unipotent and $\overline{r}|_{G_{F_v}}$ is trivial. Also we required r to be decomposed generic and enormous. We assumed that $\overline{r} \cong \overline{r}_l(\pi)$ for π satisfying various conditions, and we want to prove that r is automorphic.

We will now start proving this theorem. Choose $L/\mathbb{Q}_l$ a finite extension, $\mathcal{O} = \mathcal{O}_L$, $\mathcal{O}/\lambda = \mathbb{F}$, such that

- r is valued in $GL_n(\mathcal{O})$,
- $L \supset \tau F$ for all $\tau : F \hookrightarrow \overline{L}$,
- $\zeta_l \in L$,
- all eigenvalues of elements of $\mathrm{im}\,\overline{r}$ are in $\mathbb{F}$, and
- a certain condition for each $v \in R$ we won't specify yet (all irreducible components of a certain variety over L are geometrically irreducible).

Choose $v_0 \notin R$ such that $v_0 \nmid 2l$ and v_0 unramified in F and split in F_0 with

$$H^0(G_{F_{v_0}}, \mathrm{ad}\,\overline{r}(1)) = (0)$$

(i.e. the Frobenius element doesn't fix anything in $\mathrm{ad}\,\overline{r}(1)$; the existence of v_0 is a consequence of our condition (a) on $\overline{r}$). Let $S = R \cup \{v_0, v_0^c\}$. Let $U = \prod U_v$ be given by

$$U_{v_0} = \ker(GL_n(\mathcal{O}_{F,v_0}) \to GL_n(k(v_0)))$$

and similarly for $U_{v_0^c}$, $U_v = \mathrm{Iw}_v$ for $v \in R$, and $U_v = GL_n(\mathcal{O}_{F,v})$ for $v \notin S$. We observe (this is why we chose v_0) that U is neat. Why? If $g \in U_{v_0}$, then let $\Gamma \subset \overline{F}_{v_0}^{\times}$ be the group generated by the eigenvalues of g. We know that $g \equiv 1 \pmod{\pi_{v_0}}$. If $x \in \mathcal{O}^n_{\overline{F}_{v_0}}$ is a primitive eigenvector (one that you can't divide out by a non-unit) of g with $gx = \lambda x$, then $gx \equiv x \equiv \lambda x \pmod{\pi_{v_0}}$, so (since x is primitive) $\lambda \equiv 1 \pmod{\pi_{v_0}}$. So

$$\Gamma \subset 1 + \pi_{v_0}\mathcal{O}_{\overline{F}_{v_0}} = 1 + p\mathcal{O}_{\overline{F}_{v_0}}$$

if p lies below v_0, since v_0 is unramified. Since $p > 2$, there are no nontrivial 1-unit pth roots of unity, so $\Gamma^{tor} = \{1\}$. We conclude that g is neat, so U is neat (since you test global neatness by intersecting the local torsion subgroups). This is the only use we'll make of v_0.

Now consider the Hecke algebra $\mathbb{T}^S(U, \lambda, 1)$ and its maximal ideal

$$\mathfrak{m} = \langle \lambda, q_v^{i(i-1)/2} T_{v,i} - \mathrm{tr} \wedge^i \overline{r}(\mathrm{Frob}_v) \forall v \notin S, \forall i \rangle.$$

It's not obvious that this is a proper ideal, but if it's proper then it's maximal, since after modding out by λ we're working over a field and then we're setting every variable we have. In fact it is proper. This is because

$$(\pi^\infty \| \det \|^{(n-1)/2})^U \hookrightarrow H^\bullet(X_U, \mathscr{F}_\lambda)$$

using $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$, and $\mathbb{T}^S(U, \lambda, 1)$ acts on the source with $q_v^{i(i-1)/2} T_{v,i}$ acting as

$$\operatorname{tr} \wedge^i r_l(\pi)(\mathrm{Frob}_v)$$

by local-global compatibility for $r_l(\pi)$, giving a map

$$\begin{aligned} \mathbb{T}^S(U, \lambda, 1) &\to \mathcal{O}_{\overline{L}} \to \overline{F} \\ q_v^{i(i-1)/2} T_{v,i} &\mapsto \operatorname{tr} \wedge^i r_l(\pi)(\mathrm{Frob}_v) \end{aligned}$$

whose kernel is $\mathfrak{m}$, since $\overline{r}_l(\pi) \cong \overline{r}$.

We are next going to see that we have a surjection

$$R^{univ}_{\overline{r},S,1} \twoheadrightarrow \mathbb{T}^S(U, \lambda, 1)_{\mathfrak{m}}/I$$

where I is nilpotent, which we want the map $R^{univ}_{\overline{r},S,1} \to \overline{\mathbb{Q}}_l$ corresponding to r to factor through, and show that the kernel is nilpotent.

13 May 11: automorphy lifting—auxiliary structures.

13.1 Recap and setup

Recall that we have a CM field $F = F_0 F^+$, l large enough, $L/\mathbb{Q}_l$ finite and large enough ($\mathcal{O} = \mathcal{O}_L$), R a finite set of bad primes of F; $r : G_F \to GL_n(\mathcal{O})$ unramified outside $R \cup \{v|l\}$, crystalline above l, with "small" HT weights, with $r|_{G_{I_{F_v}}}$ unipotent for $v \in R$ and $\overline{r}|_{G_{F_v}} = 1$, and $q_v \equiv 1 \pmod{l}$; $\overline{r}$ automorphic with various conditions. We chose an auxiliary prime $v_0 \notin R$, $v \nmid 2l$, and defined $S = R \cup \{v_0, v_0^c\}$ (in order to make our open compact subgroups neat). We set $U = \prod U_v$, $U_v = \mathrm{Iw}_v$ for $v \in R$, $U_{v_0} = \ker(GL_n(\mathcal{O}_{F,v_0}) \to GL_n(k(v_0)))$, $U_{v_0^c} = \ker(\cdots)$ in the same way, $U_v = GL_n(\mathcal{O}_{F_v})$ for other v. U is neat because of the choice of U_{v_0}.

We were considering the maximal ideal of $\mathbb{T}^S(U, \lambda, 1)$ given by

$$\mathfrak{m} = \langle \lambda, q_v^{i(i-1)/2} T_{v,i} - \operatorname{tr} \wedge^i \overline{r}(\mathrm{Frob}_v) \forall v \notin S, \forall i \rangle.$$

This is a maximal ideal if proper, and it is proper because $\overline{r}$ is automorphic. Our goal is to prove that $\mathbb{T}^S(U, \lambda, 1)_{\mathfrak{m}}$ is a deformation ring.

We know that we have $r_{\mathfrak{m}} : G_F \to GL_n(\mathbb{T}^S(U, \lambda, 1)_{\mathfrak{m}}/I)$ where $I^\delta = (0)$ with $\delta = \delta([F^+ : \mathbb{Q}], n)$ depending only on $[F^+ : \mathbb{Q}]$ and n.

Let $R^{univ}_{\overline{r},S,1}$ be the universal deformation ring for lifts $r : G_F \to GL_n(A)$, where A is a complete noetherian local $\mathcal{O}$-algebra with residue field $\mathbb{F}$, of $\overline{r} : G_F \to GL_n(\mathbb{F})$, such that

- r is unramified outside $S \cup \{v|l\}$.

- r is FL at primes above l.
- for all $v \in R$, for all $\sigma \in I_{F_v}$, $\mathrm{char}_{r(\sigma)}(x) = (x-1)^n$ (i.e. $r(\sigma)$ is unipotent).

Then as usual we have a representation $r^{univ} : G_F \to GL_n(R^{univ}_{\bar{r},S,1})$ such that any such lift to A comes from conjugating the pushforward of r^{univ} along some map $R^{univ}_{\bar{r},S,1} \to A$. In particular, we have a map

$$R^{univ}_{\bar{r},S,1} \twoheadrightarrow \mathbb{T}^S(U,\lambda,1)_{\mathfrak{m}}/I$$

taking r^{univ} to $r_{\mathfrak{m}}$. (This is surjective because $T_{v,i}$ is $q_v^{-i(i-1)/2}$ times a coefficient of the characteristic polynomial of $r_{\mathfrak{m}}(\mathrm{Frob}_v)$, which will be the image of a coefficient of the characteristic polynomial of $r^{univ}(\mathrm{Frob}_v)$.) We also have a map

$$R^{univ}_{\bar{r},S,1} \to \mathcal{O}$$

taking r^{univ} to r. We would like to find a commuting map $\mathbb{T}^S(U,\lambda,1)_{\mathfrak{m}}/I \to \mathcal{O}$; this would show that r is automorphic. What we will eventually prove is that

$$\ker(R^{univ}_{\bar{r},S,1} \to \mathbb{T}^S(U,\lambda,1)_{\mathfrak{m}}/I)$$

is nilpotent, which implies that $R^{univ}_{\bar{r},S,1} \to \mathcal{O}$ factors through $\mathbb{T}^S(U,\lambda,1)_{\mathfrak{m}}/I \to \mathcal{O}$, because $\mathcal{O}$ has no nilpotent elements, so $\ker(R^{univ}_{\bar{r},S,1} \to \mathcal{O})$ contains everything nilpotent in $R^{univ}_{\bar{r},S,1}$.

13.2 Auxiliary structures

For $v \in R$, choose $\chi_{0,v} : (k(v)^\times)^n \to \mathcal{O}^\times$ of order l such that $\chi_{0,v,i} \neq \chi_{0,v,j}$ if $i \neq j$. Let $\chi_0 = \prod_{v \in R} \chi_{0,v}$. We will write χ for either χ_0 or 1 (or in many cases any l-power order character, but these are the two characters we'll be interested in). We will be interested in statements for 1, but proving things will be easier for χ_0, but because they're the same mod λ, we'll be able to transport statements to 1.

Choose Q a finite set of places of F and, for $v \in Q$, $(\alpha_{v,1}, \ldots, \alpha_{v,n}) \in (\mathbb{F}^\times)^n$ such that

1. for $v \in Q$, v is split in F_0, and Q is disjoint from $S \cup \{v|l\}$.
2. for $v \in Q$, $q_v \equiv 1 \pmod{l}$.
3. $\alpha_{v,i} \neq \alpha_{v,j}$ if $i \neq j$ (not serious, probably unnecessary).
4. for $v \in Q$, $\bar{r}(\mathrm{Frob}_v)$ has eigenvalues $\alpha_{v,1}, \ldots, \alpha_{v,n}$.

(That is, we are choosing a set of good places together with an ordering of the eigenvalues at those places.) This is called a set of Taylor-Wiles data. We attach to it a modified subgroup U_Q defined by

$$U_Q = U^Q \times \prod_{v \in Q} U_{Q,v}$$

with $\mathrm{Iw}_v \supset U_{Q,v} \supset \mathrm{Iw}_{1,v}$ such that $\mathrm{Iw}_v / U_{Q,v}$ has l-power order and $U_{Q,v}/\mathrm{Iw}_{1,v}$ has order prime to l. Let

$$U'_Q = U^Q \times \prod_{v \in Q} \mathrm{Iw}_v \trianglerighteq U_Q.$$

Let $\Delta_Q = U'_Q/U_Q = \prod_{v\in Q}\Delta_{Q,v}$, so each $\Delta_{Q,v}$ is an l-power-order abelian group. Let $\mathfrak{a}_Q \subset \mathcal{O}[\Delta_Q]$ be the augmentation ideal

$$\mathfrak{a}_Q = \langle \delta - 1 \mid \delta \in \Delta_Q\rangle$$

so that $\mathcal{O}[\Delta_Q]/\mathfrak{a}_Q = \mathcal{O}$. Let

$$\mathbb{T}_{\chi,Q} = \mathbb{T}_Q^{S\cup Q}(U'_Q/U_Q,\lambda,\chi)_{\mathfrak{n}_{\chi,Q}}$$

(recall that these are endomorphisms over $\mathcal{O}[\Delta_Q]$) where

$$\begin{aligned}\mathfrak{n}_{\chi,Q} = \langle \lambda, q_v^{i(i-1)/2}T_{v,i} - \operatorname{tr}\wedge^i\overline{r}(\mathrm{Frob}_v)\forall v\notin S\cup Q,\forall i,\\ t_{v,i,\pi_v} - \alpha_{v,i}\forall v\in Q,\forall i,\pi_v \text{ uniformizing } \mathcal{O}_{F,v}\rangle.\end{aligned}$$

(Recall that t_{v,i,π_v} is the ratio of the Hecke operator where the first i diagonal entries are π_v to the one where the first $i-1$ diagonal entries are π_v.) If $\mathfrak{n}_{\chi,Q}$ is proper, then it is maximal. For example,

$$\mathbb{T}_{1,\emptyset} = \mathbb{T}^S(U,\lambda,1)_{\mathfrak{m}}.$$

$\mathbb{T}_{\chi,Q}$ acts naturally on $C^\vee_{\chi,Q} := R\Gamma(X_{U_Q},\mathscr{F}_{\lambda,\underline{\chi}})_{U'_Q,\mathfrak{n}_{\chi,Q}} \in D^b(\mathcal{O}[\Delta_Q])$.

We are going to dualize now. (Presumably we could write down a variant of this argument without dualizing, but Richard is used to thinking about these things in the dual form.) Let

$$C_{\chi,Q} := R\operatorname{Hom}_{D(\mathcal{O}[\Delta_Q])}\left(R\Gamma(X_{U_Q},\mathscr{F}_{\lambda,\chi})_{U'_Q,\mathfrak{n}_{\chi,Q}},\mathcal{O}[\Delta_Q]\right)\left[1-n^2[F^+:\mathbb{Q}]\right].$$

Note: if A is any ring and $C\in D(A)$, we have

$$R\operatorname{Hom}_{D(A)}(R\operatorname{Hom}_{D(A)}(C,A),A)\cong C$$

(see Stacks Project [11], Theorems 15.72.3 and 15.72.2). This implies that $\mathbb{T}_{\chi,Q}$ acts faithfully on $C_{\chi,Q}$ (because it was defined as acting faithfully on its dual). If C is perfect, then $R\operatorname{Hom}_{D(A)}(C,A)$ is perfect (represented by a bounded complex of finitely generated projective elements—this is a serious restriction for non-regular rings like $\mathcal{O}[\Delta_Q]$, but is true in our setting) and

$$R\operatorname{Hom}_{D(A)}(C,A)\otimes^L_A A/I \cong R\operatorname{Hom}_{D(A/I)}(C\otimes^L_A A/I, A/I)$$

(see Exercise 10.8.4 of Weibel's book [12]). (Technically, $C\otimes^L_A A/I$ is supposed to be in $D(A)$, but you can make a similar construction that goes in $D(A/I)$. Weibel is precise and uses a different notation for this, but whatever.) Since $C^\vee_{\chi,Q} := R\Gamma(X_{U_Q},\mathscr{F}_{\lambda,\underline{\chi}})_{U'_Q,\mathfrak{n}_{\chi,Q}}$ is perfect, this implies that $C_{\chi,Q}$ is also perfect.

Now that we have the auxiliary Hecke algebras, we want the deformation-theoretic analogue. Let $R_{\chi,Q}$ be the universal deformation ring for lifts ρ of $\overline{r}$ such that

- ρ is unramified outside $S\cup\{v|l\}\cup Q$,
- ρ is FL above l,

- for all $v \in R$, $\sigma \in I_{F,v}$, $\rho(\sigma)$ has characteristic polynomial

$$\prod_i \left(X - \chi_{v,i}(\mathrm{art}_{F_v}^{-1}(\sigma))\right).$$

For example, $R_{1,\emptyset} = R^{univ}_{\bar{r},S,1}$. As before, we have a surjection

$$R_{\chi,Q} \twoheadrightarrow \mathbb{T}_{\chi,Q}/I_{\chi,Q}$$

where $I^{\delta}_{\chi,Q} = (0)$ with δ independent of χ and Q. $\mathbb{T}_{\chi,Q}/I_{\chi,Q}$ is an $\mathcal{O}[\Delta_Q]$-algebra. We need to make $R_{\chi,Q}$ an $\mathcal{O}[\Delta_Q]$-algebra in a compatible way and understand how they vary as χ and Q vary.

13.3 $\mathcal{O}[\Delta_Q]$-structure of $R_{\chi,Q}$

Lemma 13.3.1. *Let A be a complete noetherian local $\mathcal{O}$-algebra with residue field $\mathbb{F}$. Suppose we have*

$$\rho : G_{F_v} \to GL_n(A)$$

where $\overline{\rho} = \rho \pmod{\mathfrak{m}_A}$ is unramified and $\overline{\rho}(\mathrm{Frob}_v)$ has distinct eigenvalues $\alpha_1, \dots, \alpha_n \in \mathbb{F}$, and $q_v \equiv 1 \pmod{l}$. (For example, $r^{univ}_{\chi,Q}|_{G_{F_v}}$ for $v \in Q$ satisfies this.) Then

$$\rho \cong \gamma_1 \oplus \cdots \oplus \gamma_n$$

where $\gamma_i : G_{F_v} \to A^\times$ is a character and if $\varphi \in G_{F_v}$ lifts Frob_v, then $\gamma_i(\varphi) \cong \alpha_i \pmod{\mathfrak{m}_A}$. Moreover, $\gamma_i|_{I_{F_v}}$ has l-power order (this is just because $\overline{\rho}$ is unramified and $\ker(A^\times \to \mathbb{F}^\times)$ is a pro-l group). (Note that even though $\overline{\rho}$ is unramified, ρ itself is probably ramified.)

The point of this lemma is that when we relax the deformation problem to allow ramification at Q, we get more lifts, but locally at Q these lifts are still sums of characters, which can be ramified but only of l-power order, and we can identify the characters by the eigenvalues of Frob_v.

Proof. Choose $\varphi \in G_{F_v}$ lifting Frob_v. Then

$$\mathrm{char}_{\rho(\varphi)}(X) \equiv \prod (X - \alpha_i) \pmod{\mathfrak{m}_A}.$$

Since A is complete local, by Hensel's lemma, we have

$$\mathrm{char}_{\rho(\varphi)}(X) = \prod_i (X - \tilde{\alpha}_i)$$

over A where $\tilde{\alpha}_i$ lifts α_i, since we assumed the α_is were distinct. This implies that

$$\rho(\varphi) \sim \mathrm{diag}(\tilde{\alpha}_1, \dots, \tilde{\alpha}_n).$$

Why? Because we can write down idempotents

$$e_i = \frac{\prod_{j \neq i}(\rho(\varphi) - \tilde{\alpha}_j)}{\prod_{j \neq i}(\tilde{\alpha}_i - \tilde{\alpha}_j)}$$

where $\tilde{\alpha}_i - \tilde{\alpha}_j$ are units in A (since α_i, α_j are distinct mod $\mathfrak{m}_A$). We have

$$(\rho(\varphi) - \tilde{\alpha}_i)e_i = \frac{\mathrm{char}_{\rho(\varphi)}(\rho(\varphi))}{\prod_{j\neq i}(\tilde{\alpha}_i - \tilde{\alpha}_j)} = 0$$

by Cayley-Hamilton, and since $\rho(\varphi)$ acts by $\tilde{\alpha}_i$ on the image of e_i, we can write

$$e_i^2 = \frac{\prod_{j\neq i}(\tilde{\alpha}_i - \tilde{\alpha}_j)\prod_{j\neq i}(\rho(\varphi) - \tilde{\alpha}_j)}{\prod_{j\neq i}(\tilde{\alpha}_i - \tilde{\alpha}_j)\prod_{j\neq i}(\tilde{\alpha}_i - \tilde{\alpha}_j)} = e_i.$$

So e_i is an idempotent. We have $\sum e_i = 1$ because

$$\sum_i \frac{\prod_{j\neq i}(X - T_j)}{\prod_{j\neq i}(T_i - T_j)} = 1$$

over $\mathbb{Z}[X, T_1, \ldots, T_n]\left[\frac{1}{T_i - T_j}\right]_{i\neq j}$. This polynomial identity can be checked in any larger ring, for example in $\mathbb{Q}(T_1, \ldots, T_n)[X]$, where the LHS minus the RHS has degree $\leq n-1$, but is 0 at $X = T_1, \ldots, T_n$, so is identically 0.

We conclude that $A^n = \bigoplus e_i A^n$, and $\rho(\varphi)$ acts as $\tilde{\alpha}_i$ on $e_i A^n$, as we asserted. This gives us the proposition on one Frobenius lift, and now we want to extend it to the whole decomposition group.

Since $\overline{\rho}$ is unramified, $\rho(I_{F_v})$ is pro-l. Therefore ρ is trivial on wild inertia (if the prime below q_v is p, then $q_v \equiv 1 \pmod{l}$ implies $p \neq l$, and the wild inertia is pro-p). Let σ topologically generate the tame inertia. We want to show that in the basis we chose where $\rho(\varphi)$ is diagonal, $\rho(\sigma)$ is also diagonal. We are given that

$$\rho(\varphi^{-1}\sigma\varphi) = \rho(\sigma)^{q_v}.$$

We prove that $\rho(\sigma) \pmod{\mathfrak{m}_A^a}$ is diagonal by induction on a; we know this is true when $a = 1$. By the inductive hypothesis, we have $\rho(\sigma) = \Lambda + \Sigma$ where Λ is diagonal with $\Lambda \equiv \mathrm{id}_n \pmod{\mathfrak{m}_A}$ and $\Sigma \in M_{n\times n} \pmod{\mathfrak{m}_A^a}$. So the above relation becomess

$$\Lambda + \mathrm{diag}(\tilde{\alpha}_1^{-1}, \ldots, \tilde{\alpha}_n^{-1})\Sigma\,\mathrm{diag}(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n)$$

$$\equiv \Lambda^{q_v} + \Lambda^{q_v-1}\Sigma + \Lambda^{q_v-2}\Sigma\Lambda + \cdots + \Sigma\Lambda^{q_v-1} \pmod{\mathfrak{m}_A^{a+1}}$$

because any monomial in $(\Lambda + \Sigma)^{q_v}$ containing multiple Σs is $0 \pmod{\mathfrak{m}_A^{a+1}}$. This is

$$\equiv \Lambda^{q_v} + q_v\Sigma \equiv \Lambda^{q_v} + \Sigma \pmod{\mathfrak{m}_A^{a+1}}$$

since $q_v \equiv 1 \pmod{l}$. Comparing off-diagonal entries, we have when $i \neq j$ that

$$\tilde{\alpha}_i^{-1}\tilde{\alpha}_j\Sigma_{ij} \equiv \Sigma_{ij} \pmod{\mathfrak{m}_A^{a+1}}$$

but $\tilde{\alpha}_i^{-1}\tilde{\alpha}_j \not\equiv 1 \pmod{\mathfrak{m}_A}$ if $i \neq j$, so we must have $\Sigma_{ij} \in \mathfrak{m}_A^{a+1}$ if $i \neq j$, as desired.

We conclude that $\rho(\sigma)$ is diagonal, i.e. $\rho = \gamma_1 \oplus \cdots \oplus \gamma_n$, as desired (and $\gamma_i(\varphi) = \tilde{\alpha}_i$ which is $\alpha_i \pmod{\mathfrak{m}_A}$ by definition). $\square$

So we have a map

$$\begin{gathered}\mathcal{O}[\Delta_Q] \to R_{\chi,Q} \\ (\delta_1, \dots, \delta_n) \in \Delta_{Q,v} \mapsto \prod \gamma_{v,i}(\operatorname{art} \delta_i)\end{gathered}$$

where $r^{univ}_{\chi,Q}|_{G_{F_v}} = \gamma_{v,1} \oplus \cdots \oplus \gamma_{v,n}$ for $v \in Q$ as in the lemma (each $\gamma_{v,i}$ is a character of $F_v^\times$, which we can restrict to $\mathcal{O}_{F,v}^\times$, where it's tamely ramified and has l-power order, so it factors through $\Delta_{Q,v}$ and this makes sense). In a quotient of $R_{\chi,Q}$, the universal representation becomes unramified at $v \in Q$ if and only if the $\gamma_{v,i}$s become unramified, which is the same as saying that the elements of $\Delta_{Q,v}$ act trivially. Therefore $R_{\chi,Q}/\mathfrak{a}_Q$ is the maximum quotient of $R_{\chi,Q}$ on which the things in Q act trivially, that is,

$$R_{\chi,Q}/\mathfrak{a}_Q \cong R_{\chi,\emptyset}.$$

We get a surjection

$$R_{\chi,Q} \twoheadrightarrow \mathbb{T}_{\chi,Q}/I_{\chi,Q}$$

so that $r_{\chi,Q}(\varphi)$ has characteristic polynomial

$$\prod (X - \gamma_{v,i}(\varphi)) = \prod (X - t_{v,i,\pi})$$

where φ is a Frobenius lift with $\operatorname{art}^{-1} \varphi = \pi$, and we have by definition

$$\begin{aligned}\gamma_{v,i}(\varphi) &\equiv \alpha_{v,i} \pmod{\mathfrak{n}_{\chi,Q}} \\ t_{v,i,\pi} &\equiv \alpha_{v,i} \pmod{\mathfrak{n}_{\chi,Q}}\end{aligned}$$

so $\gamma_{v,i}(\varphi) = t_{v,i,\pi}$ since the $\alpha_{v,i}$s are in distinct residue classes. Since uniformizers generate $F_v^\times$, we have

$$\gamma_{v,i}(\alpha) = t_{v,i,\alpha}$$

for all $\alpha \in F_v^\times$ in $\mathbb{T}_{\chi,Q}/I_{\chi,Q}$. We conclude that the diagram

$$\begin{array}{ccc}\mathcal{O}[\Delta_\alpha] & \longrightarrow & R_{\chi,\alpha} \\ \downarrow & & \downarrow \text{surj} \\ \mathbb{T}_{\chi,Q} & \xrightarrow{\text{surj}} & \mathbb{T}_{\chi,Q}/I_{\chi,Q}\end{array}$$

commutes. Also we have $R_{\chi_0,Q}/\lambda = R_{1,Q}/\lambda$ as $\chi_0 \equiv 1 \pmod{\lambda}$ (both parametrize the deformations where something in the inertia group at a prime in R has characteristic polynomial $(X-1)^n$), and

$$r_{\chi_0,Q} \pmod{\lambda} = r_{1,Q} \pmod{\lambda}.$$

Next time, we will prove the analogues of these statements and of $R_{\chi,Q}/\mathfrak{a}_Q \cong R_{\chi,\emptyset}$ for Hecke algebras.

14 May 13: auxiliary Hecke algebra compatibilities.

14.1 Recap and setup

Recall: we had a Galois representation $r : G_F \to GL_n(\mathcal{O})$ with $\bar{r} = r \pmod{\lambda}$, and we were looking at the deformation ring $R_{\bar{r},S,1}$ and the surjection $R_{\bar{r},S,1} \twoheadrightarrow \mathbb{T}^S(U,\lambda,1)_{\mathfrak{m}}/I$. We wanted to know that the map $R_{\bar{r},S,1} \to \mathcal{O}$ corresponding to r factors through $\mathbb{T}^S(U,\lambda,1)_{\mathfrak{m}}/I$, to which end we wanted to prove that $R_{\bar{r},S,1} \twoheadrightarrow \mathbb{T}^S(U,\lambda,1)_{\mathfrak{m}}/I$ has nilpotent kernel.

We let $S = R \cup \{v_0, v_0^c\}$ for an auxiliary v_0 to satisfy technical conditions. For $v \in R$, let $\chi_v : (k(v)^\times)^n \to \mathcal{O}^\times$ have order dividing l (in practice χ_v will always be either 1 or $\chi_{0,v}$, where $\chi_{0,v}$ is some particular choice with $\chi_{0,v,i} \neq \chi_{0,v,j}$ if $i \neq j$). For $v \in Q$ ("good" and with $q_v \equiv 1 \pmod{l}$), we chose $(\alpha_{v,1}, \ldots, \alpha_{v,n})$ so that the $\{\alpha_{v,i}\}$ are eigenvalues of $\bar{r}(\mathrm{Frob}_v)$ and $\alpha_{v,i} \neq \alpha_{v,j}$ for $i \neq j$.

We chose $U_Q \trianglelefteq U'_Q$ so that $U_Q^Q = (U'_Q)^Q = U^Q$, and for $v \in Q$, $U'_{Q,v} = \mathrm{Iw}_v$ and $\mathrm{Iw}_v \supset U_{Q,v} \supset \mathrm{Iw}_{v,1}$ is such that the first containment has l-power index and the second containment has index prime to l. Then $U'_Q/U_Q = \Delta_Q$ where Δ_Q is a finite abelian group of l-power order; this has $(\#Q)n$ generators. We let $\mathfrak{a}_Q \subset \mathcal{O}[\Delta_Q]$ be the augmentation ideal.

Let $R_{\chi,Q}$ be the universal deformation ring of lifts of $\bar{r}$ that

- are unramified outside $S \cup Q$,
- are FL above L, and
- satisfy: for $v \in R$, $\sigma \in I_{F_v}$, $\rho(\sigma)$ has characteristic polynomial

$$\prod_{i=1}^{n}(X - \chi_{v,i}(\mathrm{art}^{-1}\sigma)).$$

We defined a map $\mathcal{O}[\Delta_Q] \to R_{\chi,Q}$ and saw that

- $R_{\chi,Q}/\mathfrak{a}_Q = R_{\chi,\emptyset}$
- $R_{1,\emptyset} = R_{\bar{r},S,1}$
- $R_{\chi,Q}/\lambda \cong R_{\chi',Q}/\lambda$ as $\mathbb{F}[\Delta_Q]$-algebras, because all the characters reduce to $1 \pmod{\lambda}$.

We defined the $\mathcal{O}[\Delta_Q]$-algebra $\mathbb{T}_{\chi,Q} = \mathbb{T}_Q^{Q\cup S}(U'_Q/U_Q,\lambda,\chi)_{\mathfrak{n}_{\chi,Q}}$, acting on the perfect complex $C^\vee_{\chi,Q} = R\Gamma(X_{U_Q}, \mathscr{F}_{\lambda,\chi})_{U'_Q,\mathfrak{n}_{\chi,Q}} \in D^b(\mathcal{O}[\Delta_Q])$ (where we can pick out the part corresponding to $\mathfrak{n}_{\chi,Q}$ because the derived category is "idempotent complete"), and also acting on the dual

$$C_{\chi,Q} = R\,\mathrm{Hom}_{D(\mathcal{O}[\Delta_Q])}\left(C^\vee_{\chi,Q}, \mathcal{O}[\Delta_Q]\right)\left[1 - n^2[F^+ : \mathbb{Q}]\right].$$

(Note that the generators we wrote down for $\mathfrak{n}_{\chi,Q}$ last time don't really depend on χ, it is just in an algebra that depends on χ.) We have $\mathbb{T}_{1,\emptyset} = \mathbb{T}^S(U,\lambda,1)_{\mathfrak{m}}$ and

$$R_{\chi,Q} \twoheadrightarrow \mathbb{T}_{\chi,Q}/I_{\chi,Q}$$

where $I_{\chi,Q}^{\delta} = (0)$ with δ independent of χ, Q. We want to explain the analogue of the above facts about $R_{\chi,Q}$ for $\mathbb{T}_{\chi,Q}$. We again have

$$C_{\chi,Q}^{\vee} \otimes_{\mathcal{O}[\Delta_Q]}^{L} \mathbb{F}[\Delta_Q] \cong C_{\chi',Q}^{\vee} \otimes_{\mathcal{O}[\Delta_Q]}^{L} \mathbb{F}[\Delta_Q]$$

because both are $R\Gamma(X_{U_Q}, \mathscr{F}_{\lambda,\chi} \otimes_{\mathcal{O}} \mathbb{F})_{U'_Q,\mathfrak{n}_{Q,\chi}}$ and everything in here reduces to the same thing over $\mathbb{F}$. Similarly,

$$C_{\chi,Q} \otimes_{\mathcal{O}[\Delta_Q]}^{L} \mathbb{F}[\Delta_Q] \cong C_{\chi',Q} \otimes_{\mathcal{O}[\Delta_Q]}^{L} \mathbb{F}[\Delta_Q].$$

We get a surjection $\mathbb{T}_{\chi,Q}/\lambda \twoheadrightarrow \overline{\mathbb{T}}_{\chi,Q}$, where $\overline{\mathbb{T}}_{\chi,Q}$ is defined to be the quotient of $\mathbb{T}_{\chi,Q}/\lambda$ that acts faithfully on the LHS; we similarly write $\mathbb{T}_{\chi,Q}/\lambda \twoheadrightarrow \overline{\mathbb{T}}_{\chi',Q}$, and we have

$$\overline{\mathbb{T}}_{\chi,Q} \cong \overline{\mathbb{T}}_{\chi',Q}.$$

We can write maps

$$R_{\chi,Q}/\lambda \twoheadrightarrow \overline{\mathbb{T}}_{\chi,Q}/I_{\chi,Q} \twoheadleftarrow \overline{\mathbb{T}}_{\chi,Q} \twoheadrightarrow \overline{\mathbb{T}}_{\chi,Q}/(I_{\chi,Q} + I_{\chi',Q})$$

$$R_{\chi',Q}/\lambda \twoheadrightarrow \overline{\mathbb{T}}_{\chi',Q}/I_{\chi',Q} \twoheadleftarrow \overline{\mathbb{T}}_{\chi',Q} \twoheadrightarrow \overline{\mathbb{T}}_{\chi',Q}/(I_{\chi,Q} + I_{\chi',Q})$$

where the first, third, and fourth terms in the two rows are isomorphic (the fourth terms make sense because the isomorphism between the third terms means that $I_{\chi,Q}$ and $I_{\chi',Q}$ can both be viewed as ideals in either $\overline{\mathbb{T}}_{\chi,Q}$ or $\overline{\mathbb{T}}_{\chi',Q}$). We would have liked to say that the second terms are isomorphic, but we don't really know what's in the two nilpotent ideals (they're not even uniquely defined). But we have additional surjections

$$\overline{\mathbb{T}}_{\chi,Q}/I_{\chi,Q} \twoheadrightarrow \overline{\mathbb{T}}_{\chi,Q}/(I_{\chi,Q} + I_{\chi',Q})$$

$$\overline{\mathbb{T}}_{\chi',Q}/I_{\chi',Q} \twoheadrightarrow \overline{\mathbb{T}}_{\chi',Q}/(I_{\chi,Q} + I_{\chi',Q})$$

and we claim that all of these surjections and isomorphisms commute with each other. This is clear when you unravel the definitions.

14.2 Statement and proof of Hecke algebra compatibility

Proposition 14.2.1. *1. $R\Gamma(\Delta_Q, C_{\chi,Q}^{\vee}) \cong C_{\chi,\emptyset}^{\vee}$, giving a surjection*

$$\mathbb{T}_{\chi,Q}/\mathfrak{a}_Q \twoheadrightarrow \mathbb{T}_{\chi,\emptyset}$$

and, dually,

$$C_{\chi,Q} \otimes_{\mathcal{O}[\Delta_Q]}^{L} \mathcal{O} \cong C_{\chi,\emptyset}$$

(this is implied by the previous isomorphism and the fact that $C_{\chi,Q}^{\vee}$ is perfect).

2. Various compatibilities hold: we know that

$$(C_{\chi',Q} \otimes_{\mathcal{O}[\Delta_Q]} \mathbb{F}[\Delta_Q]) \otimes_{\mathbb{F}[\Delta_Q]} \mathbb{F} \cong (C_{\chi,Q} \otimes_{\mathcal{O}[\Delta_Q]} \mathbb{F}[\Delta_Q]) \otimes_{\mathbb{F}[\Delta_Q]} \mathbb{F}$$

but also

$$\begin{aligned}(C_{\chi,Q}\otimes_{\mathcal{O}[\Delta_Q]}\mathbb{F}[\Delta_Q])\otimes_{\mathbb{F}[\Delta_Q]}\mathbb{F}&\cong(C_{\chi,Q}\otimes_{\mathcal{O}[\Delta_Q]}\mathcal{O})\otimes_{\mathcal{O}}\mathbb{F}\\(C_{\chi',Q}\otimes_{\mathcal{O}[\Delta_Q]}\mathbb{F}[\Delta_Q])\otimes_{\mathbb{F}[\Delta_Q]}\mathbb{F}&\cong(C_{\chi',Q}\otimes_{\mathcal{O}[\Delta_Q]}\mathcal{O})\otimes_{\mathcal{O}}\mathbb{F}\\(C_{\chi,Q}\otimes_{\mathcal{O}[\Delta_Q]}\mathcal{O})\otimes_{\mathcal{O}}\mathbb{F}&\cong C_{\chi,\emptyset}\otimes^L_{\mathcal{O}}\mathbb{F}\\(C_{\chi',Q}\otimes_{\mathcal{O}[\Delta_Q]}\mathcal{O})\otimes_{\mathcal{O}}\mathbb{F}&\cong C_{\chi',\emptyset}\otimes^L_{\mathcal{O}}\mathbb{F}\\C_{\chi,\emptyset}\otimes^L_{\mathcal{O}}\mathbb{F}&\cong C_{\chi',\emptyset}\otimes^L_{\mathcal{O}}\mathbb{F},\end{aligned}$$

and all of these isomorphisms commute.

Part 2 follows easily from Part 1: you can construct the isomorphisms in Part 1 over $\mathbb{F}$ as well as $\mathcal{O}$, and over $\mathbb{F}$ the constructions for χ and χ' have the same meaning. So we just need to prove the first assertion in Part 1.

Proof. We have

$$R\Gamma(\Delta_Q, R\Gamma(X_{U_Q},\mathscr{F}_{\lambda,\chi})_{U'_Q})\cong R\Gamma(X_{U'_Q},\mathscr{F}_{\lambda,\chi})$$

equivariantly for the Hecke operators $T_{v,i}$ for $v\notin S\cup Q$, and $t_{v,i,\alpha}$ for $v\in Q$, $\alpha\in F_v^\times$. (This isn't obvious, but we did check it.) Therefore, by localizing,

$$R\Gamma(\Delta_Q, C^\vee_{\chi,Q})\cong R\Gamma(X_{U'_Q},\mathscr{F}_{\lambda,\chi})_{\mathfrak{n}_{Q,\chi,0}}$$

where $\mathbb{T}_{\chi,Q}$ acts on the left, compatibly with the isomorphic Hecke algebra $\mathbb{T}(U'_Q,\lambda,\chi)_{\mathfrak{n}_{Q,\chi,0}}$ acting on the right, where $\mathfrak{n}_{Q,\chi,0}\trianglelefteq\mathbb{T}(U'_Q,\lambda,\chi)$ is the ideal defined by the same relations as before in the new Hecke algebra:

$$\langle\lambda; T_{v,i}q_v^{i(i-1)/2}-\operatorname{tr}\wedge^i\overline{r}(\mathrm{Frob}_v), v\notin Q\cup S; t_{v,i,\pi}-\alpha_{v,i}, v\in Q, \pi \text{ uniformizing } F_v\rangle.$$

The LHS is the complex we are interested in. We want to compare it to $C^\vee_{\chi,\emptyset}$ with its action of $\mathbb{T}_{\chi,\emptyset}$, but this does not look like the RHS, because at primes in Q the Hecke operators aren't the same. At primes in Q, the Hecke operators in $\mathbb{T}(U'_Q,\lambda,\chi)_{\mathfrak{n}_{Q,\chi,0}}$ are those from the Iwahori-Hecke algebra (with the relation $t_{v,i,\pi}-\alpha_{v,i}$), whereas the ones in $\mathbb{T}_{\chi,\emptyset}$ are from the unramified Hecke algebra (with the relation $T_{v,i}q_v^{i(i-1)/2}-\operatorname{tr}\wedge^i\overline{r}(\mathrm{Frob}_v)$). So we need to compare these. We have

$$R\Gamma(X_{U'_Q},\mathscr{F}_{\lambda,\chi})\xrightarrow{\operatorname{tr}_{U'_Q/U_\emptyset}}R\Gamma(X_{U_\emptyset},\mathscr{F}_{\lambda,\chi})$$

$$R\Gamma(X_{U'_Q},\mathscr{F}_{\lambda,\chi})\xleftarrow{i_{U'_Q/U_\emptyset}}R\Gamma(X_{U_\emptyset},\mathscr{F}_{\lambda,\chi})$$

and these commute with $T_{v,i}$ for $v\notin S\cup Q$. We know that

$$\operatorname{tr}_{U'_Q/U_\emptyset}\circ i_{U'_Q/U_\emptyset}=[U_\emptyset:U'_Q]\equiv(n!)^{\#Q}\pmod{\lambda}$$

where $n!\in\mathcal{O}^\times$, so this is an isomorphism. We also claim that

$$q_v^{j(j-1)/2}T_{v,j}\circ\operatorname{tr}_{U'_Q/U_\emptyset}=\operatorname{tr}_{U'_Q/U_\emptyset}\circ S_j(t_{v,1},\ldots,t_{v,n})$$

where $t_{v,i} = t_{v,i,\pi_v}$ and S_j is the jth elementary symmetric function. This is because we previously saw that

$$\begin{aligned}\operatorname{tr}_{U'_Q/U_\emptyset} \circ \operatorname{tr}_v \circ S_j(t_{v,1}, \dots, t_{v,n}) &= \operatorname{tr}_{U'_Q/U_\emptyset} \circ i(q_v^{j(j-1)/2} T_{v,j})[GL_n(\mathcal{O}_{F,v}) : \mathrm{Iw}_v] \\ &= q_v^{j(j-1)/2} T_{v,j} [GL_n(\mathcal{O}_{F_v}) : \mathrm{Iw}_v] \operatorname{tr}_{U'_Q/U_\emptyset}\end{aligned}$$

so we would like to know that

$$[GL_n(\mathcal{O}_{F,v}) : \mathrm{Iw}_v] \operatorname{tr}_{U'_Q/U_\emptyset} = \operatorname{tr}_{U'_Q/U_\emptyset} \circ \operatorname{tr}_v$$

but this is true from the same equality for $j = 0$. Since $[GL_n(\mathcal{O}_{F_v}) : \mathrm{Iw}_v]$ is a unit, cancelling it out gives us the claim. Similarly

$$i_{U'_Q/U_\emptyset} q_v^{j(j-1)/2} T_{v,j} = S_j(t_{v,1}, \dots, t_{v,n}) i_{U'_Q/U_\emptyset}.$$

If $w \in S_n$ and $v \in Q$, we have

$$\begin{aligned}S_{v,w} \circ i_{U'_Q/U_\emptyset} &= q_v^{\mathrm{length}(w)} i_{U'_Q/U_\emptyset} \\ \operatorname{tr}_{U'_Q/U_\emptyset} \circ S_{v,w} &= q_v^{\mathrm{length}(w)} \operatorname{tr}_{U'_Q/U_\emptyset} \\ \operatorname{tr}_{U'_Q/U_\emptyset} \circ i_{U'_Q/U_\emptyset} &= \prod_{v \in Q} \operatorname{tr}_v .\end{aligned}$$

These all have similar proofs.

Now inside $\mathbb{T}_Q^{Q \cup S}(U'_Q, \lambda, \chi)$ we have $\tilde{\mathbb{T}}$, the $\mathcal{O}$-subalgebra generated by $T_{v,j}$ for $v \notin Q \cup S$ and $S_j(t_{v,1}, \dots, t_{v,n})$ for $v \in Q$. Let $\tilde{\mathfrak{m}} \trianglelefteq \tilde{\mathbb{T}}$ be defined by the same thing as before:

$$\langle \lambda; T_{v,j} q_v^{j(j-1)/2} - \operatorname{tr} \wedge^j \overline{r}(\mathrm{Frob}_v), v \notin S \cup Q; S_j(t_{v,1}, \dots, t_{v,n}) - \operatorname{tr} \wedge^j \overline{r}(\mathrm{Frob}_v), v \in Q \rangle.$$

Now the formulas we wrote down tell us that the action of $\tilde{\mathbb{T}}$ is compatible with the Hecke action at lower level; that is, because the Hecke operators match up,

$$R\Gamma(X_{U'_Q}, \mathscr{F}_{\lambda,\chi})_{\tilde{\mathfrak{m}}} \xrightarrow{\operatorname{tr}_{U'_Q/U_\emptyset}} R\Gamma(X_{U_\emptyset}, \mathscr{F}_{\lambda,\chi})_{\mathfrak{n}_{\emptyset,\chi}}$$

$$R\Gamma(X_{U'_Q}, \mathscr{F}_{\lambda,\chi})_{\tilde{\mathfrak{m}}} \xleftarrow{i_{U'_Q/U_\emptyset}} R\Gamma(X_{U_\emptyset}, \mathscr{F}_{\lambda,\chi})_{\mathfrak{n}_{\emptyset,\chi}}$$

are compatible with the actions of $\tilde{\mathbb{T}}_{\tilde{\mathfrak{m}}}$ on the left and $\mathbb{T}_{\chi,\emptyset}$ on the right, and the surjection

$$\tilde{\mathbb{T}}_{\tilde{\mathfrak{m}}} \twoheadrightarrow \mathbb{T}_{\chi,\emptyset}$$

(this is a surjection because the composite of $\operatorname{tr}_{U'_Q/U_\emptyset}$ and $i_{U'_Q/U_\emptyset}$ is a unit multiple of the identity, so $i_{U'_Q/U_\emptyset}$ is an inclusion).

Since $\mathbb{T}(U'_Q, \lambda, \chi)_{\tilde{\mathfrak{m}}}$ also acts on $R\Gamma(U'_Q, \mathscr{F}_{\lambda,\chi})_{\tilde{\mathfrak{m}}}$, we get a map

$$\tilde{\mathbb{T}}_{\tilde{\mathfrak{m}}} \to \mathbb{T}(U'_Q, \lambda, \chi)_{\tilde{\mathfrak{m}}}$$

which is finite because we are just putting in a finite number of additional $t_{v,i}$s, which are roots of monic polynomials with coefficients in $\tilde{\mathbb{T}}_{\tilde{\mathfrak{m}}}$. Since a finite algebra over a complete noetherian local ring is semilocal, we can decompose

$$\mathbb{T}(U'_Q, \lambda, \chi)_{\tilde{\mathfrak{m}}} = \prod_{\mathfrak{n} \text{ over } \tilde{\mathfrak{m}}} \mathbb{T}(U'_Q, \lambda, \chi)_{\mathfrak{n}}$$

where the $\mathfrak{n}$s over $\tilde{\mathfrak{m}}$ are in bijection with permutations $\sigma = (\sigma_v) \in S_n^Q$ via

$$\mathfrak{n}_\sigma = \langle \lambda; T_{v,j} q_v^{j(j-1)/2} - \operatorname{tr} \wedge^j \overline{r}(\mathrm{Frob}_v), v \notin S \cup Q; t_{v,j} - \alpha_{v,\sigma_v j}, v \in Q \rangle.$$

(That is, since the $t_{v,j}$s are the roots of polynomials whose roots are the $\alpha_{v,j}$s, they must be those numbers up to some order.) Therefore the LHS can also be written

$$\bigoplus_{\sigma \in S_n^Q} R\Gamma(X_{U'_Q}, \mathscr{F}_{\lambda,\chi})_{\mathfrak{n}_\sigma} \xrightarrow{\operatorname{tr}_{U'_Q/U_\emptyset}} R\Gamma(U_\emptyset, \mathscr{F}_{\lambda,\chi})_{\mathfrak{n}_{\emptyset,\chi}}$$

$$\bigoplus_{\sigma \in S_n^Q} R\Gamma(X_{U'_Q}, \mathscr{F}_{\lambda,\chi})_{\mathfrak{n}_\sigma} \xleftarrow{i_{U'_Q/U_\emptyset}} R\Gamma(U_\emptyset, \mathscr{F}_{\lambda,\chi})_{\mathfrak{n}_{\emptyset,\chi}}.$$

Also $\tilde{\mathbb{T}}_{\tilde{\mathfrak{m}}} \twoheadrightarrow \mathbb{T}(U'_Q, \lambda, \chi)_{\mathfrak{n}_\sigma}$ for all σ, because in the target we added the roots of some polynomials with distinct roots, so by Hensel's Lemma the roots already exist in the source. So we're really getting repeated Hecke operators, not more Hecke operators. We claim that in fact

$$\operatorname{tr}_{U'_Q/U_\emptyset} : R\Gamma(X_{U'_Q}, \mathscr{F}_{\lambda,\chi})_{\mathfrak{n}_\sigma} \xrightarrow{\sim} R\Gamma(U_\emptyset, \mathscr{F}_{\lambda,\chi})_{\mathfrak{n}_{\emptyset,\chi}}$$

is an isomorphism for any σ; once we prove this we're done.

Proof: it suffices to prove this after taking $\otimes^L_{\mathcal{O}} \mathbb{F}$ because of abstract stuff about derived categories. That is, if we have $f : A \to B$ in $D^b(\mathcal{O})$, where $H^\bullet(A)$, $H^\bullet(B)$ are finitely generated over $\mathcal{O}$, and extend it to a distinguished triangle $A \xrightarrow{f} B \to C \to$, f is an isomorphism if and only if $C = 0$. If we then take $A \otimes^L_{\mathcal{O}} \mathbb{F} \xrightarrow{f} B \otimes^L_{\mathcal{O}} \mathbb{F} \to C \otimes^L_{\mathcal{O}} \mathbb{F} \to$, under our assumptions, C also has finitely generated cohomology, so we have that $f \otimes \mathbb{F}$ is an isomorphism if and only if $C \otimes^L \mathbb{F} \cong (0)$. So we want to see that $C \cong^L (0)$ if and only if $C \otimes^L \mathbb{F} \cong (0)$. But since we're in cohomological dimension 1, in $D(\mathcal{O})$ we have

$$C \cong \bigoplus_i H^i(C)[-i] \cong \bigoplus \mathcal{O}[-i]/x$$

because any finitely generated $\mathcal{O}$-module is a direct sum of cyclic modules. So we need to show that $\mathcal{O}/x \cong (0)$ in $D(\mathcal{O})$ iff $\mathcal{O}/x \otimes^L \mathbb{F} \cong (0)$ in $D(\mathbb{F})$. But

$$\mathcal{O}/x \otimes^L \mathbb{F} = \mathcal{O}/x \otimes (\mathcal{O}[-1] \xrightarrow{\pi_\lambda} \mathcal{O}[0]) = \mathcal{O}/x[-1] \xrightarrow{\pi_\lambda} \mathcal{O}/x[0]$$

which has cohomology $(\mathcal{O}/x)[\pi_\lambda]$ in degree -1 and $\mathcal{O}/(x, \pi_\lambda)$ in degree 0. $\mathcal{O}/(x, \pi_\lambda)$ is 0 iff x is a unit, which is true iff $\mathcal{O}/x = (0)$, so we are done.

Okay so we have

$$R\Gamma(X_{U'_Q}, \mathscr{F}_{\lambda,\chi})_{\mathfrak{n}_\sigma} \otimes^L \mathbb{F} \xrightarrow{\operatorname{tr}} R\Gamma(X_{U_\emptyset}, \mathscr{F}_{\lambda,\chi})_{\mathfrak{n}_{\chi,\emptyset}} \otimes^L \mathbb{F}$$

and writing proj_σ for the projection onto the σ-component and incl_σ for the inclusion of the σ-component, this becomes

$$R\Gamma(X_{U'_Q}, \mathscr{F}_{\lambda,\chi} \otimes_{\mathcal{O}} \mathbb{F})_{\mathfrak{n}_\sigma} \xrightarrow{\mathrm{tr}_{U'_Q/U_\emptyset} \circ \mathrm{incl}_\sigma} R\Gamma(X_{U_\emptyset}, \mathscr{F}_{\lambda,\chi} \otimes_{\mathcal{O}} \mathbb{F})_{\mathfrak{n}_{\chi,\emptyset}}$$

$$R\Gamma(X_{U'_Q}, \mathscr{F}_{\lambda,\chi} \otimes_{\mathcal{O}} \mathbb{F})_{\mathfrak{n}_\sigma} \xleftarrow{\mathrm{proj}_\sigma \circ \mathrm{tr}_{U'_Q/U_\emptyset}} R\Gamma(X_{U_\emptyset}, \mathscr{F}_{\lambda,\chi} \otimes_{\mathcal{O}} \mathbb{F})_{\mathfrak{n}_{\chi,\emptyset}}$$

and we claim that these are mutually inverse (at least $\pmod{\lambda}$). In one direction, since

$$i_{U'_Q/U_\emptyset} \circ \mathrm{tr}_{U'_Q/U_\emptyset} = \mathrm{tr} = \sum_{\sigma' \in S_n^Q} \sigma',$$

and $\sigma' \circ \mathrm{incl}_\sigma = \mathrm{incl}_{\sigma'}$, we have

$$\mathrm{proj}_\sigma \circ i_{U'_Q/U_\emptyset} \circ \mathrm{tr}_{U'_Q/U_\emptyset} \circ \mathrm{incl}_\sigma = \mathrm{proj}_\sigma \circ \sum_{\sigma'} \mathrm{incl}_{\sigma'} = \mathrm{id}_\sigma .$$

In the other direction, we claim that

$$\mathrm{tr}_{U'_Q/U_\emptyset} \circ \mathrm{incl}_\sigma \circ \mathrm{proj}_\sigma \circ i_{U'_Q/U_\emptyset}$$

is independent of σ, because for any τ we have $\mathrm{tr}_{U'_Q/U_\emptyset} = \mathrm{tr}_{U'_Q/U_\emptyset} \circ \tau$, so

$$\mathrm{tr}_{U'_Q/U_\emptyset} \circ \mathrm{incl}_\sigma \circ \mathrm{proj}_\sigma \circ i_{U'_Q/U_\emptyset} = \mathrm{tr}_{U'_Q/U_\emptyset} \circ \tau \circ \mathrm{incl}_\sigma \circ \mathrm{proj}_\sigma \circ i_{U'_Q/U_\emptyset}$$
$$\mathrm{tr}_{U'_Q/U_\emptyset} \circ \mathrm{incl}_{\sigma\tau^{-1}} \circ \mathrm{proj}_{\sigma\tau^{-1}} \circ \tau \circ i_{U'_Q/U_\emptyset}$$

which is the thing we started with but for $\sigma\tau^{-1}$. Summing over σ, we get

$$\sum_\sigma \mathrm{tr}_{U'_Q/U_\emptyset} \circ \mathrm{incl}_\sigma \circ \mathrm{proj}_\sigma \circ i_{U'_Q/U_\emptyset} = \mathrm{tr}_{U'_Q/U_\emptyset} \circ i_{U'_Q/U_\emptyset} = [U_\emptyset : U'_Q]$$

but also, because all the summands are the same,

$$\sum_\sigma \mathrm{tr}_{U'_Q/U_\emptyset} \circ \mathrm{incl}_\sigma \circ \mathrm{proj}_\sigma \circ i_{U'_Q/U_\emptyset} = (n!)^{\#Q}\, \mathrm{tr}_{U'_Q/U_\emptyset} \circ \mathrm{incl}_\sigma \circ \mathrm{proj}_\sigma \circ i_{U'_Q/U_\emptyset}$$

which is what we wanted, because we said earlier that $(n!)^{\#Q} \equiv [U_\emptyset : U'_Q] \pmod{\lambda}$. □

This is pretty much all we need to know about automorphic forms; next time we'll start proving things.

15 May 18: framed deformation rings.

15.1 Review

We have an l-adic representation of the absolute Galois group of a CM field: $r : G_F \to GL_n(\mathcal{O})$ where $\mathcal{O} = \mathcal{O}_L$ with $L/\mathbb{Q}_l$ finite and F is CM. We want to prove that it is automorphic, assuming that $\overline{r} : G_F \to GL_n(\mathbb{F})$ is automorphic. We assume r is unramified outside

$R \cup \{v|l\}$ where R is a fixed finite set and arrange by base change that $q_v \equiv 1 \pmod{l}$ for $v \in R$. We chose a pair of auxiliary primes $\{v_0, v_0^c\}$ and let $S = R \cup \{v_0, v_0^c\}$.

We consider varying finite sets Q of auxiliary primes v such that v is "good" (unramified etc.), $q_v \equiv 1 \pmod{l}$, and $\overline{r}(\mathrm{Frob}_v)$ has distinct eigenvalues (this is unfortunately restrictive). We also consider varying characters $\chi : \prod_{v \in R} (k(v)^\times)^n \to \mathcal{O}^\times$ of l-power order (here there are only finitely many choices, and in fact we will only care about two of them, including 1).

For each of these choices, we wrote $R_{\chi,Q}$ for the universal deformation ring for lifts of $\overline{r}$ that are unramified outside $S \cup \{v|l\} \cup Q$ and Fontaine-Laffaile above l, and such that for $v \in R$ and $\sigma \in I_{F_v}$, $\rho(\sigma)$ has characteristic polynomial $\prod_i (X - \chi_{v,i}(\mathrm{art}^{-1} \sigma))$. We saw that if Δ_Q is the maximal l-power quotient of $\prod_{v \in Q} (k(v)^\times)^n$, we have a map $\mathcal{O}[\Delta_Q] \to R_{\chi,Q}$, because the universal representation on the decomposition group above a prime in Q can be diagonalized into tamely ramified characters which factor through $(k(v)^\times)^n$. We wrote $\mathfrak{a}_Q$ for the augmentation ideal of Δ_Q (which upon being modded out takes all elements of Δ_Q to 1).

We saw that $R_{\chi,Q}/\mathfrak{a}_Q \cong R_{\chi,\emptyset}$ and $R_{\chi,Q}/\lambda \cong R_{\chi',Q}/\lambda$ for any χ, χ'. These two isomorphisms are compatible: if you take the first isomorphism for χ and reduce it mod λ, then use the second isomorphism for Q and $\emptyset$, you get back the first isomorphism mod λ for χ'.

On the automorphic side, we defined a perfect complex $C_{\chi,Q} \in D^b(\mathcal{O}[\Delta_Q])$. We will need to know how many terms we need to represent this complex. For this, we should look at

$$H^\bullet \left(C_{\chi,Q} \otimes^L_{\mathcal{O}[\Delta_Q]} \mathbb{F} \right)$$

(where the map $\mathcal{O}[\Delta_Q] \to \mathbb{F}$ is by modding out both $\mathfrak{a}_Q$ and λ). This is supported in a range of $\bullet$ independent of χ and Q (given by the dimension of the locally symmetric space). We would like a stronger statement than that, but we only know that after inverting l:

$$H^\bullet \left(C_{\chi,Q} \otimes^L_{\mathcal{O}[\Delta_Q]} L \right) \neq (0)$$

only for $\bullet \in [q_0, q_0 + l_0]$, where $q_0 = [F^+ : \mathbb{Q}] n(n-1)/2$ and $l_0 = [F^+ : \mathbb{Q}] n - 1$. (We expect the same statement to be true for $\mathbb{F}$ in place of L—which would have stronger consequences—but have no idea how to prove it.)

We also know that $H^\bullet(C_{1,\emptyset} \otimes_{\mathcal{O}} L) \neq (0)$. This follows from the fact that $\overline{r}$ is automorphic, because that means there is a RA cuspidal automorphic representation π_0 such that $\overline{r_{\pi_0}} \cong \overline{r}$. Suppose $\pi_{0,\infty}$ has $HC(\pi_{0,\infty}) = -HT(\rho)$ where ρ is an algebraic representation of $\mathrm{res}^F_{\mathbb{Q}} GL_n$ and $\mathfrak{m} \trianglelefteq \mathbb{T}$ is such that $\overline{r_{\mathfrak{m}}} \cong \overline{r}$. Then we claim that

$$H^\bullet(X, \mathscr{L}_\rho)_{\mathfrak{m}} \otimes \mathbb{Q}_l \neq (0)$$

which after choosing $\mathbb{Q}_l \hookrightarrow \mathbb{C}$ follows from

$$H^\bullet(X, \mathscr{L}_\rho)_{\mathfrak{m}} \otimes \mathbb{C} \neq (0)$$

but the LHS is

$$\bigoplus_{\pi} (\pi^\infty)^U_{\mathfrak{m}} \otimes H^\bullet(\mathfrak{g}, U_\infty, \rho \otimes \pi_\infty).$$

Then the term of this sum associated to π_0 is nonzero because $HC(\pi_{0,\infty}) = -HT(\rho)$ implies that $H^\bullet(\mathfrak{g}, U_\infty, \rho \otimes \pi_{0,\infty}) \neq (0)$, and $\overline{r_{\mathfrak{m}}} \cong \overline{r_{\pi_0}}$ implies that $(\pi_0^\infty)^U_{\mathfrak{m}} \neq (0)$. (The latter statement can be proven as follows: if $T \in \mathbb{T}$, then T acts on π_0^U by $\pi(T) \in L \subset \overline{\mathbb{Q}}_l \cong \mathbb{C}$. We need to check that if $T \notin \mathfrak{m}$ then $\pi_0(T) \neq 0$. But $\pi_0(T) \pmod{l} = T \pmod{\mathfrak{m}} \neq 0$ because the possible Ts are generated by things of the form $\mathrm{tr} \wedge^i r_m(\mathrm{Frob}_v)$, and the possible $\pi_0(T)$s are generated by things of the form $\mathrm{tr} \wedge^i r_{\pi_0}(\mathrm{Frob}_v)$, and $\overline{r_{\pi_0}} = \overline{r_{\mathfrak{m}}}$.)

We described the action of a Hecke algebra $\mathbb{T}_{\chi,Q}$ on $C_{\chi,Q}$, where we again have a (finite) map $\mathcal{O}[\Delta_Q] \to \mathbb{T}_{\chi,Q}$, and we are given a surjection $R_{\chi,Q} \twoheadrightarrow \mathbb{T}_{\chi,Q}/I_{\chi,Q}$ where $I_{\chi,Q}^\delta = (0)$ for δ independent of χ and Q (we expect $I_{\chi,Q}$ to be 0 but have no idea how to prove that either).

Our goal is to prove that the kernel of $R_{1,\emptyset} \twoheadrightarrow \mathbb{T}_{1,\emptyset}/I_{1,\emptyset}$ is nilpotent.

The claim that $C_{\chi,Q}$ is perfect and has cohomology with support in a small range is the fundamental input we need. What that tells us is that $C_{\chi,Q}$ is "large": if you have a complex of projectives of small length, they can't cancel out much. For example, a complex of projectives of length 1 would just give rise to one module that's projective over $\mathcal{O}[\Delta_Q]$; since Δ_Q gets arbitrarily large, the complex is also "large". "As you allow more ramification, the size of the space of automorphic forms grows significantly."

We gave an isomorphism $C_{\chi,Q} \otimes^L_{\mathcal{O}[\Delta_Q]} \mathcal{O} \xrightarrow{\sim} C_{\chi,\emptyset}$ and a surjection $\mathbb{T}_{\chi,Q}/\mathfrak{a}_Q \twoheadrightarrow \mathbb{T}_{\chi,\emptyset}$ compatible with $R_{\chi,Q}/\mathfrak{a}_Q \twoheadrightarrow R_{\chi,Q}$, where by compatible we mean that the two maps

$$R_{\chi,Q}/\mathfrak{a}_Q \twoheadrightarrow \mathbb{T}_{\chi,Q}/(I_{\chi,Q}, \mathfrak{a}_Q) \twoheadrightarrow \mathbb{T}_{\chi,\emptyset}/I_{\chi,Q} \twoheadrightarrow \mathbb{T}_{\chi,\emptyset}/(I_{\chi,\emptyset} + I_{\chi,Q})$$
$$R_{\chi,Q}/\mathfrak{a}_Q \xrightarrow{\sim} R_{\chi,\emptyset} \twoheadrightarrow \mathbb{T}_{\chi,\emptyset}/I_{\chi,\emptyset} \twoheadrightarrow \mathbb{T}_{\chi,\emptyset}/(I_{\chi,\emptyset} + I_{\chi,Q})$$

are the same. We also have an isomorphism

$$C_{\chi,Q} \otimes^L_{\mathcal{O}[\Delta_Q]} \mathbb{F}[\Delta_Q] \xrightarrow{\sim} C_{\chi',Q} \otimes^L_{\mathcal{O}[\Delta_Q]} \mathbb{F}[\Delta_Q]$$

for any χ, χ', again because these are cohomologies of locally symmetric spaces with coefficients mod λ and χ, χ' are the same mod λ. We have actions of $\mathbb{T}_{\chi,Q}/\lambda$ on the source and $\mathbb{T}_{\chi',Q}/\lambda$ on the target, with respective faithfully acting quotients $\overline{\mathbb{T}}_{\chi,Q}$ and $\overline{\mathbb{T}}_{\chi',Q}$; then

$$\overline{\mathbb{T}}_{\chi,Q} \xrightarrow{\sim} \overline{\mathbb{T}}_{\chi',Q}$$

again compatibly with $R_{\chi,Q}/\lambda \cong R_{\chi',Q}/\lambda$ in the same sense as before (after taking a common nilpotent quotient). Finally, $C_{\chi,Q} \otimes^L_{\mathcal{O}[\Delta_Q]} \mathbb{F}[\Delta_Q] \xrightarrow{\sim} C_{\chi',Q} \otimes^L_{\mathcal{O}[\Delta_Q]} \mathbb{F}[\Delta_Q]$ and $C_{\chi,Q} \otimes^L_{\mathcal{O}[\Delta_Q]} \mathcal{O} \xrightarrow{\sim} C_{\chi,\emptyset}$ are compatible: if you mod the second isomorphism for χ out by λ and replace χ with χ' using the first isomorphism, you get back the second isomorphism for χ'.

In some sense, what we have said is all we need to know about automorphic forms: as long we have complexes of modules $C_{\chi,Q}$ with the properties we have stated, $R_{1,\emptyset} \to \mathbb{T}_{1,\emptyset}/I_{1,\emptyset}$ is essentially an isomorphism. This is because to know that your space of automorphic forms gives all Galois representations, it is enough to show that the space of automorphic forms "grows as fast as the Galois deformation rings can grow" when you add tame ramification.

15.2 Framed deformations

We can fix $r^{univ}_{\chi,Q} : G_F \to GL_n(R_{\chi,Q})$ a representative of the universal deformation such that

$$r^{univ}_{\chi,Q} \pmod{\lambda} = r^{univ}_{\chi',Q} \pmod{\lambda}$$
$$r^{univ}_{\chi,Q} \pmod{\mathfrak{a}_Q} = r^{univ}_{\chi,\emptyset}$$

as follows.

- First make a choice of $r^{univ}_{1,\emptyset}$.

- Then make compatible choices of $r^{univ}_{1,Q}$ for all Q: if you pick some universal representation and reduce it mod $\mathfrak{a}_Q$ you get a conjugate of $r^{univ}_{1,\emptyset}$, so you can take the thing conjugating it, lift it to $R_{1,Q}$, and conjugate by the inverse.

- Then make compatible choices of all $r^{univ}_{\chi,\emptyset}$ for all χ in the same way, since again you just have one reduction, this time mod λ.

- Then finally make compatible choices of all $r^{univ}_{\chi,Q}$. For this, we need two compatibilities: we have a map
$$R_{\chi,Q} \to R_{\chi,\emptyset} \times_{R_{1,\emptyset}/\lambda} R_{1,Q}/\lambda \cong R_{\chi,Q}/(\mathfrak{a}_Q \cap \lambda)$$
and the composite $R_{\chi,Q} \twoheadrightarrow R_{\chi,Q}/(\mathfrak{a}_Q \cap \lambda)$ is a surjection, so we can lift $r^{univ}_{\chi,\emptyset} \times (r_{1,Q} \pmod{\lambda})$ (the factors of which are indeed compatible in $R_{1,Q}/\lambda$) to $R_{\chi,Q}$.

Let $\mathscr{T} = \mathcal{O}[\![X_1, \dots, X_{n^2\#(S\cup\{w|l\})-1}]\!]$. The problem with universal deformation rings is that they don't exist locally because the image of the Galois representation is too small; we need to work with universal lifting rings, where we don't mod out by conjugations. But then we don't get a map to the global case, so we need framed deformation rings.

Let $R^{\square}_{\chi,Q}$ be the universal framed deformation ring, representing the functor

$$\{\text{complete noetherian local } \mathcal{O}\text{-algebras with residue field } \mathbb{F}\} \to \mathit{Sets}$$

taking A to equivalence classes of tuples $(\rho, \{\alpha_v\}_{v\in S\cup\{w|l\}})$ where

- $\rho : G_F \to GL_n(A)$ is a lift of $\bar{r}$ such that
  - ρ is unramified outside $S \cup Q \cup \{w|l\}$,
  - ρ is FL above l, and
  - for $\sigma \in I_{F_v}$, $v \in R$, we have $\mathrm{char}_{\rho(\sigma)}(x) = \prod(X - \chi_{v,i} \circ \mathrm{art}^{-1}\sigma)$; and
- $\alpha_v \in I_n + M_{n\times n}(\mathfrak{m}_A)$ for each v,

and the equivalence relations are $(\rho, \{\alpha_v\}) \sim (g\rho g^{-1}, \{g\alpha_v\})$ for $g \in I_n + M_{n\times n}(\mathfrak{m}_A)$. Over $R_{\chi,Q}$ there was no canonical universal representation, but over $R^{\square}_{\chi,Q}$ there is a canonical lifting of $\bar{r}|_{G_{F_v}}$ for $v \in S \cup \{w|l\}$, i.e. $\alpha_v^{-1}\rho\alpha_v|_{G_{F_v}}$.

How does this relate to the universal deformation ring? Our choice of $r^{univ}_{Q,\chi}$ gives an isomorphism
$$R^{\square}_{\chi,Q} \cong R_{\chi,Q}\widehat{\otimes}_{\mathcal{O}}\mathscr{T}$$
with the universal tuple being $\left(r^{univ}_{Q,\chi}, \{I_n + (X_{v,ij})\}_{v\in S\cup\{w|l\}}\right)$, where $X_{v,ij}$ is one of the variables from $\mathscr{T}$ except $X_{v_0,nn}$, which is 0. Why? We have

$$Z_{GL_n(R_{\chi,Q})}(r^{univ}_{Q,\chi}) = R^{\times}_{\chi,Q}$$

since $\bar{r}$ is absolutely irreducible, by a version of Schur's lemma. So

$$(r^{univ}_{Q,\chi}, \{\alpha_v\}) \sim (r^{univ}_{Q,\chi}, \{\beta_v\})$$

if and only if there is $\mu \in \mathfrak{m}_A$ such that $\beta_v = (1+\mu)\alpha_v$ for all $v \in S \cup \{w|l\}$. But in particular multiplying by $1+\mu$ would take

$$1 + X_{v_0,nn} \mapsto (1+\mu)(1 + X_{v_0,nn}) = 1 + X_{v_0,nn} + \mu(1 + X_{v_0,nn})$$

and hence take $X_{v_0,nn}$ to $X_{v_0,nn} + \mu(1 + X_{v_0,nn})$, so requiring $X_{v_0,nn} = 0$ means $\mu = 0$. So $\left(r^{univ}_{Q,\chi}, \{I_n + (X_{v,ij})\}_{v \in S \cup \{w|l\}}\right)$ is indeed the desired universal tuple.

This isomorphism is compatible with

$$R^{\square}_{\chi,Q}/\mathfrak{a}_Q \cong R^{\square}_{\chi,Q}$$
$$R^{\square}_{\chi,Q}/\lambda \cong R^{\square}_{\chi',Q}/\lambda.$$

15.3 Local lifting rings, part 1

For $v = v_0, v_0^c$, let $R^{\square}_v$ be the universal lifting ring for $\overline{r}|_{G_{F_v}}$, without equivalence by conjugation relation.

Proposition 15.3.1. *This is isomorphic to a power series ring in n^2 variables over $\mathcal{O}$, because $H^0(G_{F_v}, (\operatorname{ad}\overline{r})(1)) = (0)$ (which we assumed when we picked v_0).*

Proof. First, we claim that

$$\dim_{\mathbb{F}} \mathfrak{m}_{R^{\square}_v}/(\mathfrak{m}^2_{R^{\square}_v}, \lambda) = n^2$$

and consequently $R^{\square}_v$ can be topologically generated by n^2 variables over $\mathcal{O}$. This is because

$$\operatorname{Hom}(\mathfrak{m}_{R^{\square}_v}/(\mathfrak{m}^2_{R^{\square}_v}, \lambda), \mathbb{F}) = \operatorname{Hom}_{\mathcal{O}-alg}(R^{\square}_v, \mathbb{F}[\epsilon]/(\epsilon^2))$$

because to determine an element of the RHS, we know what it does on $\mathcal{O}$, so we just need to say what it does on $\mathfrak{m}_{R^{\square}_v}$; since the square of the maximal ideal is 0 in $\mathbb{F}[\epsilon]/(\epsilon^2)$, it must take $\mathfrak{m}^2_{R^{\square}_v}$ to 0; since λ is 0 in $\mathbb{F}[\epsilon]/(\epsilon^2)$, it must take λ to 0; therefore the map on $\mathfrak{m}_{R^{\square}_v}$ factors through the $\mathbb{F}$-vector space $\mathfrak{m}_{R^{\square}_v}/(\mathfrak{m}^2_{R^{\square}_v}$; and any such $\mathbb{F}$-linear map from the LHS gives rise to an element of the RHS by the correspondence

$$f \mapsto (a \mapsto (a \pmod{\mathfrak{m}_{R^{\square}_v}}) + \epsilon(f(a - (\text{any lift of } \mathfrak{a} \pmod{\mathfrak{m}_{R^{\square}_v}}) \text{ to } \mathcal{O}))).$$

Now, $\operatorname{Hom}_{\mathcal{O}-alg}(R^{\square}_v, \mathbb{F}[\epsilon]/(\epsilon^2))$ is the set of deformations of $\overline{r}$ to $\mathbb{F}[\epsilon]/(\epsilon^2)$. Such a deformation is given by $\overline{r}(1+\epsilon\varphi)$ for some $\varphi : G_F \to M_{n\times n}(\mathbb{F})$, which is a homomorphism if and only if $\varphi \in Z^1(G_{F_v}, \operatorname{ad}\overline{r})$. Consequently

$$\dim_{\mathbb{F}} \mathfrak{m}_{R^{\square}_v}/(\mathfrak{m}^2_{R^{\square}_v}, \lambda) = \dim_{\mathbb{F}} Z^1(G_{F_v}, \operatorname{ad}\overline{r}).$$

To compute this dimension, we look at the exact sequence

$$0 \to H^0(G_{F_v}, \operatorname{ad}\overline{r}) \to \operatorname{ad}\overline{r} \to Z^1(G_{F_v}, \operatorname{ad}\overline{r}) \twoheadrightarrow H^1(G_{F_v}, \operatorname{ad}\overline{r}) \to 0$$

so

$$\dim_{\mathbb{F}} Z^1(G_{F_v}, \operatorname{ad}\overline{r}) = \dim H^1(G_{F_v}, \operatorname{ad}\overline{r}) + \dim \operatorname{ad}\overline{r} - \dim H^0(G_{F_v}, \operatorname{ad}\overline{r})$$

where $\dim \mathrm{ad}\,\overline{r} = n^2$. By the Tate local Euler characteristic formula for $v \nmid l$, the above becomes

$$\dim H^2(G_{F_v}, \mathrm{ad}\,\overline{r}) + n^2$$

and by Tate local duality and the fact that $\mathrm{ad}\,\overline{r}$ is self-dual under the trace pairing $(x, y) \mapsto \mathrm{tr}(xy)$, this is

$$H^0(G_{F_v}, \mathrm{ad}\,\overline{r}(1))^\vee + n^2.$$

We assumed that $H^0(G_{F_v}, \mathrm{ad}\,\overline{r}(1))^\vee = 0$, so this is just n^2, as desired (otherwise it could be bigger).

We conclude that

$$\mathcal{O}[\![T_1, \ldots, T_{n^2}]\!] \twoheadrightarrow R_v^\square$$

is an isomorphism on tangent spaces, so if J is the kernel, we have $J \subset (T_1, \ldots, T_{n^2})^2$, and the map factors as

$$\mathcal{O}[\![T_1, \ldots, T_{n^2}]\!] \twoheadrightarrow \mathcal{O}[\![T_1, \ldots, T_{n^2}]\!]/J\mathfrak{m} \twoheadrightarrow R_v^\square.$$

The universal lifting $r^\square$ lifts to $\tilde{r} : G_{F_v} \to GL_n(\mathcal{O}[\![T_1, \ldots, T_n]\!]/J\mathfrak{m})$, because if you lift the representation one group element at a time and try to check whether the result is still a representation, its failure to be one turns out to be measured by a 2-cocycle, which has to be a 2-boundary because we assumed that $H^2(G_{F_v}, \mathrm{ad}\,\overline{r}) = (0)$. Then you can alter your chosen liftings element-wise and get an actual representation. By the universal property of $R_v^\square$, we get a map in the other direction

$$R_v^\square \to \mathcal{O}[\![T_1, \ldots, T_{n^2}]\!]/J\mathfrak{m}$$

giving rise to $\tilde{r}$, such that the composite of the maps in the two directions is the identity on $R_v^\square$. This splitting means we can decompose

$$\mathcal{O}[\![T_1, \ldots, T_{n^2}]\!]/J\mathfrak{m} \cong R_v^\square \oplus J/\mathfrak{m}J$$

where $J/\mathfrak{m}J$ has square zero and $R_v^\square$ acts via $R_v^\square \twoheadrightarrow \mathbb{F}$. On tangent spaces this isomorphism becomes

$$(T_1, \ldots, T_{n^2})/(\lambda + (T_1, \ldots, T_{n^2})^2) \cong \mathfrak{m}_{R_v^\square}/(\mathfrak{m}_{R_v^\square}^2, \lambda) \oplus J/\mathfrak{m}J$$

but the LHS has dimension n^2 and so does $\mathfrak{m}_{R_v^\square}/(\mathfrak{m}_{R_v^\square}^2, \lambda)$, so $J/\mathfrak{m}J$ has dimension 0, so $J/\mathfrak{m}J = (0)$, so $J = (0)$ by Nakayama's lemma. □

This is the kind of argument we make a lot: first, you control the tangent space by calculating H^1 using its interpretation as a space of lifts, which tells you how many generators you need, and then you control the number of relations in terms of H^2 (for example in this case there are no relations because we assumed H^2 to vanish).

15.4 Local lifting rings, part 2

For $v|l$, let $R_v^\square$ be the universal Fontaine-Laffaile lifting ring of $\overline{r}|_{G_{F_v}}$. This is standard to compute—you can do it by deforming Fontaine-Laffaile modules rather than Galois representations, and FL modules are linear algebra objects and easy to compute with. What you get is a power series ring in

$$n^2 + \frac{n(n-1)}{2}[F_v : \mathbb{Q}_l]$$

variables over $\mathcal{O}$.

For $v \in R$, let $R^{\square}_{v,\chi_v}$ be the universal lifting ring of $\overline{r}|_{G_{F_v}}$ such that $\rho(\sigma)$ has characteristic polynomial $\prod_{i=1}^n (X - \chi_{v,i}(\mathrm{art}^{-1}\sigma))$ for all $\sigma \in I_{F_v}$. The structure of this is more complicated, and depends on χ_v, but will be crucial for the arguments. For now, we won't prove anything.

The idea is that we again have a linear algebra problem: the representation is tamely ramified, hence determined by the image of Frobenius and the image of a generator of tame inertia. So what we really have is a moduli space of a pair of matrices with some commutation relation and some restriction on the characteristic polynomial of one of the matrices. The conclusion is that we need to understand the deformation theory of a pair of matrices Φ, Σ such that

$$\Phi\Sigma\Phi^{-1} = \Sigma^{q_v}$$
$$\mathrm{char}_\Sigma(X) = \prod_{i=1}^n (X - \zeta_i)$$

where $\zeta_i = \chi_{v,i}$(generator of tame inertia), a fixed l-power roof of unity.

Proposition 15.4.1. *We may choose L large enough that for all (finitely many!) $\underline{\chi}$ and all minimal prime ideals $\mathfrak{p}$ of $R^{\square}_{v,\chi_v}$ (resp. minimal prime ideals of $R^{\square}_{v,\chi_v}/\lambda$), the ring $R^{\square}_{v,\chi_v}/\mathfrak{p}$ is geometrically integral, meaning that $R^{\square}_{v,\chi_v}/\mathfrak{p} \otimes_{\mathcal{O}} \mathcal{O}_{L'}$ is integral for all L'/L finite (resp. $R^{\square}_{v,\chi_v}/(\lambda, \mathfrak{p})$ is geometrically integral over $\mathbb{F}$). [Note: Richard thinks the following statement is actually true over any L, but hasn't seen a proof of that. Instead, we should absorb this into the conditions on L we put at the very beginning.] Then*

1. *If the $\chi_{v,i}$ are all distinct as i varies for a fixed v, then $R^{\square}_{v,\chi_v}$ is geometrically irreducible (in fact $R^{\square}_{v,\chi_v}$ mod the unique minimal prime ideal is geometrically integral) and has (Krull) dimension $1+n^2$. (This is relative dimension n^2 over $\mathcal{O}$, which is the same as what happened for v_0, v_0^c. This is typical of what happens when you stay away from l.)*

2. *$R^{\square}_{v,1}$ is equidimensional of dimension $1+n^2$ and every generic point has characteristic 0 (there are no irreducible components only in characteristic l). Moreover, every generic point of $R^{\square}_{v,1}/\lambda$ is the specialization of a unique generic point of $R^{\square}_{v,1}$ (and similarly every generic point of $R^{\square}_{v,1}$ specializes to a unique generic point of $R^{\square}_{v,1}/\lambda$, though we won't need that).*

In summary, if all the characters are 1, the deformation space has many components, but they look the same in characteristic 0 as they do in characteristic l. If all the characters are different, then it is irreducible in characteristic 0, but since its reduction mod λ is the same as if all the characters are 1, so the special fiber must have lots of components.

Next time, we will discuss the map

$$\widehat{\bigotimes_v} R^{\square}_{v,\chi_v} \to R^{\square}_{\chi,Q}.$$

16 May 20: dimension of auxiliary deformation rings.

16.1 Structure of local lifting rings summary

Last time, we introduced for each $v \in S \cup \{w|l\}$ a local lifting ring $R^{\square}_v$.

- For $v = v_0, v_0^c$, we took all liftings, getting a power series ring in n^2 variables over $\mathcal{O}$.
- For $v|l$, we took FL liftings, getting a power series ring in $n^2 + \frac{n(n-1)}{2}[F_v : \mathbb{Q}_l]$ variables over $\mathcal{O}$.
- For $v \in R$, and $\chi_v : (k(v)^\times)^n \to \mathcal{O}^\times$ of l-power order (in practice either $\chi_v = 1$ or $\chi_{0,v}$ where $\chi_{0,v,i} \neq \chi_{0,v,j}$ for all $i \neq j$), we defined $R^\square_{v,\chi_v}$, and said that it has equidimension n^2+1, every irreducible component is geometrically irreducible, and every generic point has characteristic 0.
  - Furthermore, in the case of $\chi_{0,v}$, we said that $R^\square_{v,\chi_v}$ is geometrically irreducible (in the nonstandard sense that it remains irreducible upon $\otimes_\mathcal{O}\mathcal{O}_{L'}$ for all L'/L finite).
  - In the case of 1, we said that every irreducible component of $R^\square_{v,1}/\lambda$ is the specialization of a unique irreducible component of $R^\square_{v,1}$.

Let

$$R^{loc}_\chi = \widehat{\bigotimes}_{v \in S \cup \{w|l\}, \mathcal{O}} R^\square_{v,(\chi_v)}$$

(where the (χ_v) only appears if $v \in R$).

Lemma 16.1.1. *1. If A and B are complete noetherian local $\mathbb{F}$-algebras with residue field $\mathbb{F}$, and if the reduced irreducible components of $\operatorname{spec} A$ and $\operatorname{spec} B$ are geometrically integral (i.e. $A/\mathfrak{p}$ remains integral upon $\otimes_\mathbb{F}\mathbb{F}'$ for any $\mathbb{F}'/\mathbb{F}$ finite), then the irreducible components of $A\widehat{\otimes}_\mathbb{F}B$ are in bijection with pairs of an irreducible component of A and one of B, with $(\mathfrak{p} \in \operatorname{spec} A, \mathfrak{q} \in \operatorname{spec} B)$ corresponding to $\langle \mathfrak{p}, \mathfrak{q}\rangle \trianglelefteq A\widehat{\otimes}_\mathbb{F}B$. Moreover, $A\widehat{\otimes}_\mathbb{F}B/\langle \mathfrak{p}, \mathfrak{q}\rangle$ is geometrically irreducible and has dimension $\dim A/\mathfrak{p} + \dim B/\mathfrak{q}$. (This is what you expect to be true for schemes of finite type over $\mathbb{F}$ with the normal tensor product; we are saying it remains true for complete noetherian local $\mathbb{F}$-algebras with the completed tensor product.)*

2. Suppose A, B are complete noetherian local $\mathcal{O}$-algebras with residue field $\mathbb{F}$ and that the reduced irreducible components of $\operatorname{spec} A$ and $\operatorname{spec} B$ are flat over $\mathcal{O}$ and geometrically integral (in the nonstandard integral sense previously described). Then the irreducible components of $\operatorname{spec} A\widehat{\otimes}_\mathcal{O}B$ are in bijection with pairs of an irreducible component of A and one of B, with $(\mathfrak{p} \in \operatorname{spec} A, \mathfrak{q} \in \operatorname{spec} B)$ corresponding to $\langle \mathfrak{p}, \mathfrak{q}\rangle \trianglelefteq \operatorname{spec} A\widehat{\otimes}_\mathcal{O}B$. Moreover, $A\widehat{\otimes}_\mathcal{O}B/\langle \mathfrak{p}, \mathfrak{q}\rangle$ is flat over $\mathcal{O}$, geometrically integral, and has (Krull) dimension $\dim A/\mathfrak{p} + \dim B/\mathfrak{q} - 1$. (The -1 is because you expect the relative dimensions over $\mathcal{O}$ to add, so you have to subtract 1 from $\dim A/\mathfrak{p}$, $\dim B/\mathfrak{q}$, add them, and add 1 back.)

In the case of $R^{loc}_{\underline{\chi}}$, we know that $R^{loc}_{\underline{\chi}}/\lambda$ is independent of χ. We also know that for $\chi = \chi_0, 1$, $R^{loc}_{\underline{\chi}}$ is equidimensional of dimension

$$1 + n^2\#(S \cup \{w|l\}) + [F : \mathbb{Q}]\frac{n(n-1)}{2}$$

and every generic point has characteristic 0. $R^{loc}_{\chi_0}$ is irreducible. (Actually geometrically irreducible, but we don't really care about that anymore: if you take the product of two schemes which are irreducible but not geometrically irreducible, you can't expect the product to remain irreducible—for example, if you take the product of spec of two non-linearly disjoint extensions of $\mathbb{Q}$, then you get spec of a tensor product of two fields over a common subfield, which can split into a product of fields. So we needed to know geometric irreducibility to pass irreducibility to the product, but now that we've already taken products we only need irreducibility from here on.) On the other hand, every irreducible component of R^{loc}_1/λ is the specialization of a unique irreducible component of R^{loc}_1.

16.2 Dimension of framed deformation rings

We have a map $R^{loc}_\chi \to R^\square_{\chi,Q}$ as follows. For $v \in S \cup \{w|l\}$ we have a representation $\alpha_v^{-1} r_{\chi,Q}|_{G_{F_v}} \alpha_v$, which gives a map $R^{loc}_{v,\chi_v} \to R^\square_{\chi,Q}$, and we take the product of those. We want to study $R^\square_{\chi,Q}$ as an algebra over R^{loc}_χ to calculate its relative tangent space.

Lemma 16.2.1. *$R^\square_{\chi,Q}$ can be topologically generated over R^{loc}_χ by*

$$\dim H^1_{\mathscr{L}_Q^\perp}(G_F, (\operatorname{ad}\overline{r})(1)) + n\#Q - [F^+ : Q]n^2$$

elements, where $\mathscr{L}_Q^\perp$ is the Selmer group condition given by the classes in H^1 that are unramified outside $S \cup \{w|l\}$ and trivial above Q (more detail in proof).

Proof. We need to calculate $\dim_{\mathbb{F}} \mathfrak{m}_{R^\square_{\chi,Q}} / \left(\mathfrak{m}^2_{R^\square_{\chi,Q}}, \mathfrak{m}_{R^{loc}_\chi}\right)$; then a basis for this $\mathbb{F}$-vector space lifted to $R^\square_{\chi,Q}$ will topologically generate it over R^{loc}_χ. The dual is

$$\left\{R^\square_{\chi,Q} \to \mathbb{F}[\epsilon]/(\epsilon^2) \mid R^{loc}_\chi \to R^\square_{\chi,Q} \to \mathbb{F}[\epsilon]/(\epsilon^2) \text{ factors through } \mathbb{F}\right\}.$$

(The argument is the same as last time: the dual of just $\mathfrak{m}_{R^\square_{\chi,Q}} / \left(\mathfrak{m}^2_{R^\square_{\chi,Q}}, \lambda\right)$ is given by the algebra homomorphisms $R^\square_{\chi,Q} \to \mathbb{F}[\epsilon]/(\epsilon^2)$, because to give such a map we know what happens to $\mathcal{O}$ so we just need to say what happens on $\mathfrak{m}_{R^\square_{\chi,Q}}$, but those elements must go to multiples of ϵ, which have square 0, so such a map must kill $\mathfrak{m}^2_{R^\square_{\chi,Q}}$ and also λ. But this time we're also requiring the map to be trivial on $\mathfrak{m}_{R^{loc}_\chi}$, so its restriction to R^{loc}_χ must factor through $\mathbb{F}$.) This dual is the same as tuples

$$\left((1+\varphi\epsilon)\overline{r}, (\operatorname{id}_n + a_v\epsilon)_{v \in S \cup \{w|l\}}\right)$$

where $(1+\varphi\epsilon)\overline{r}$ gives a lifting of $\overline{r}$ to $\mathbb{F}[\epsilon]/(\epsilon^2)$ and $(\operatorname{id}_n + a_v\epsilon)_{v \in S \cup \{w|l\}}$ gives the framing matrices, so we need

$$\varphi : G_{F, S \cup \{v|l\} \cup Q} \to M_{n \times n}(\mathbb{F})$$

to satisfy

$$\varphi(\sigma\tau) = \varphi(\sigma) + \operatorname{ad}\overline{r}(\sigma)\varphi(\tau)$$

so that $(1+\varphi\epsilon)\overline{r}$ is a homomorphism; furthermore, for all $v \in S \cup \{w|l\}$, we need to conjugate $(1+\varphi\epsilon)\overline{r}$ by $\operatorname{id}_n + a_v\epsilon$ and have the resulting map out of $R^\square_{v,(\chi_v)}$ factor through $\mathbb{F}$, which turns out to mean requiring that

$$\varphi|_{G_{F_v}} + (\operatorname{ad}\overline{r} - 1)a_v = 0.$$

These tuples should be taken up to equivalence, where

$$(\varphi, (a_v)) \sim (\varphi + (1 - \operatorname{ad} \overline{r})a, (a_v + a)), \qquad a \in M_{n\times n}(\mathbb{F})$$

Now this space can be rephrased as the space of tuples $(\varphi, (a_v))$ such that

$$\varphi \in Z^1(G_{F,S\cup\{w|l\}\cup Q}, \operatorname{ad} \overline{r}), \qquad a_v \in M_{n\times n}(\mathbb{F})$$

with $\varphi|_{G_{F_v}} = (1 - \operatorname{ad} \overline{r})a_v$ for all $v \in S \cup \{w|l\}$, modded out by $M_{n\times n}(\mathbb{F})$, included into this space via the map

$$a \mapsto \left((1 - \operatorname{ad} \overline{r})a, (a)_{v\in S\cup\{w|l\}}\right).$$

This space surjects onto

$$\ker\left(H^1(G_{F,S\cup\{w|l\}\cup Q}, \operatorname{ad} \overline{r}) \to \bigoplus_{v\in S\cup\{w|l\}} H^1(G_{F_v}, \operatorname{ad} \overline{r})\right)$$

via $(\varphi, (a_v)) \mapsto [\varphi]$, since we require φ to be trivial at places in $S\cup\{w|l\}$. (This is a surjection because any class in $H^1(G_{F,S\cup\{w|l\}\cup Q}, \operatorname{ad} \overline{r})$ can be written as a cocycle, and if it is in the kernel then it restricts to a coboundary in each $H^1(G_{F_v}, \operatorname{ad} \overline{r})$, so we can find a corresponding a_v.)

What is the kernel of $(\varphi, (a_v)) \mapsto [\varphi]$? Tuples in which φ is a coboundary, that is, those of the form

$$\left\{((1 - \operatorname{ad} \overline{r})a, (a_v)) \mid a, a_v \in M_{n\times n}(\mathbb{F}), a - a_v \in (\operatorname{ad} \overline{r})^{G_{F_v}}\right\}/M_{n\times n}(\mathbb{F})$$

where $M_{n\times n}(\mathbb{F})$ is included via $b \mapsto ((1-\operatorname{ad} \overline{r})b, (b))$ for $b \in M_{n\times n}(\mathbb{F})$. (Despite the notation, we are just remembering $(1-\operatorname{ad} \overline{r})a$, not a itself—we're just saying that some such a exists.) The dimension of this is

$$n^2 - \dim H^0(G_F, \operatorname{ad} \overline{r}) + \sum_{v\in S\cup\{w|l\}} H^0(G_{F_v}, \operatorname{ad} \overline{r}) - n^2$$

where n^2 is the contribution from the initial possible choices of a, the $-\dim H^0(G_F, \operatorname{ad} \overline{r})$ is to subtract off those as for which $(1-\operatorname{ad} \overline{r})0$, the $+\sum_{v\in S\cup\{w|l\}} H^0(G_{F_v}, \operatorname{ad} \overline{r})$ is the contribution from the possible choices of each a_v, and the $-n^2$ is to subtract off the possible choices of b, because $b \mapsto ((1 - \operatorname{ad} \overline{r})b, (b))$ is injective as long as $S \cup \{w|l\}$ is nonempty. So we just get

$$\sum_{v\in S\cup\{w|l\}} H^0(G_{F_v}, \operatorname{ad} \overline{r}) - \dim H^0(G_F, \operatorname{ad} \overline{r}).$$

So the dimension we are trying to compute is

$$\dim H^1_{\mathscr{L}_Q}(G_F, \operatorname{ad} \overline{r}) + \sum_{v\in S\cup\{w|l\}} H^0(G_{F_v}, \operatorname{ad} \overline{r}) - \dim H^0(G_F, \operatorname{ad} \overline{r}).$$

If $\mathscr{L} = \{L_v\}$ with $L_v \subset H^1(G_{F_v}, M)$, $L_v = H^1(G_{F_v}/I_{F_v}, M^{I_{F_v}})$ for almost all v, recall that

$$H^1_{\mathscr{L}}(G_F, M) = \ker\left(H^1(G_F, M) \to \bigoplus_v H^1(F_v, M)/L_v\right).$$

We write $\mathscr{L}^\perp = \{\mathscr{L}_v^\perp\}$ where $L_v^\perp$ is the annihilator of L_v under the local Tate duality pairing

$$H^1(G_{F_v}, M) \times H^1(G_{F_v}, M^\vee(1)) \to \mathbb{Q}/\mathbb{Z}$$

(for M of finite cardinality; we could also do vector spaces over fields of characteristic 0 but the target of the pairing would be $\mathbb{Q}_l$ or whatever). The condition we gave in the statement was $\mathscr{L}_Q = \{L_{Q,v}\}$ where

$$L_{Q,v} = \begin{cases} (0) & v \in S \cup \{w|l\} \\ H^1(G_{F_v}, \operatorname{ad}\overline{r}) & v \in Q \\ H^1(G_{F_v}/I_{F_v}, \operatorname{ad}\overline{r}) & v \notin Q \cup S \cup \{w|l\}. \end{cases}$$

(In the last line we don't have to put $(\operatorname{ad}\overline{r})^{I_{F_v}}$ because $\operatorname{ad}\overline{r}$ is assumed to be unramified for such v.)

To simplify our expression, we need to use Tate global duality to relate Selmer groups to dual Selmer groups. The following formulation of global duality was written down by Wiles, with previous work by Greenberg. Let M be over $\mathbb{F}$. The formula says that

$$\dim H^1_{\mathscr{L}}(G_F, M) - \dim H^1_{\mathscr{L}^\perp}(G_F, M^\vee(1))$$
$$= \dim H^0(G_F, M) - \dim H^0(G_F, M^\vee(1)) + \sum_v (\dim L_v - \dim H^0(G_{F_v}, M))$$

where the $\sum_v$ is over all places, including ones at ∞, but also the terms of the sum are 0 for almost all v. (This formula actually works for any finite cardinality module if you replace dim by order and write everything multiplicatively.) In our case, since $\operatorname{tr} : \operatorname{ad}\overline{r} \times \operatorname{ad}\overline{r} \to \mathbb{F}$, $(x, y) \mapsto \operatorname{tr}(xy)$ is an equivariant perfect duality and so $\operatorname{ad}\overline{r}$ is self-dual, we get

$$\begin{aligned}\dim H^1_{\mathscr{L}_Q^\perp}(G_F, \operatorname{ad}\overline{r}(1)) &+ \dim H^0(G_F, \operatorname{ad}\overline{r}) - \dim H^0(G_F, (\operatorname{ad}\overline{r})(1)) - \dim H^0(G_F, \operatorname{ad}\overline{r}) \\ &+ \sum_{v \in S \cup \{w|l\}} \big((\dim L_{Q,v} = 0) - \dim H^0(G_{F_v}, \operatorname{ad}\overline{r}) + \dim H^0(G_{F_v}, \operatorname{ad}\overline{r})\big) \\ &+ \sum_{v \in Q} \big(\dim H^1(G_{F_v}, \operatorname{ad}\overline{r}) - \dim H^0(G_{F_v}, \operatorname{ad}\overline{r})\big) \\ &+ \sum_{v|\infty} \big(0 - (\dim H^0(G_{F_v}, \operatorname{ad}\overline{r}) = n^2)\big).\end{aligned}$$

The dimension in the last sum is n^2 because we are in a complex CM field, so every place is a complex place, so the local Galois group is trivial, so $H^0(G_{F_v}, \operatorname{ad}\overline{r}) = \operatorname{ad}\overline{r}$. In the second-to-last sum, by local duality, we get

$$\dim H^1(G_{F_v}, \operatorname{ad}\overline{r}) - \dim H^0(G_{F_v}, \operatorname{ad}\overline{r}) = \dim H^2(G_{F_v}, \operatorname{ad}\overline{r}) = \dim H^0(G_{F_v}, \operatorname{ad}\overline{r}(1)) = n.$$

The last equality is because at primes in Q, $\overline{r}$ is unramified, so we just need to look at Frob_v; the order of the residue field is $1 \pmod{l}$, so the cyclotomic character takes Frob_v to 1, so we can drop the (1); now Frob_v has distinct eigenvalues $\alpha_{v,1}, \ldots, \alpha_{v,n}$, and its eigenvalues on $\operatorname{ad}\overline{r}$ are ratios of pairs of these, which will be 1 exactly when you take ratios of $\alpha_{v,i}$ with itself; there are n such eigenvalues.

Finally, since we assumed that $\operatorname{ad}\overline{r}|_{G_{F(\zeta_l)}}$ is irreducible, we have

$$H^0(G_F, (\operatorname{ad}\overline{r})(1)) = H^0(G_F, \mathbb{F}(1)) = (0)$$

since again we can drop the (1) in the first expression after including ζ_l, after which by Schur's lemma the only invariants in $\operatorname{ad}\overline{r}$ come from multiplication by a scalar, so we get the second expression; since $\zeta_l \notin F$, the cyclotomic character is nontrivial and $\mathbb{F}(1)$ has no invariants.

Everything adds up to

$$\dim H^1_{\mathscr{L}_Q^\perp}(G_F, (\operatorname{ad}\overline{r})(1)) + n\#Q - n^2[F^+ : \mathbb{Q}]$$

as desired. □

We will see that for suitable choices of Q, we can make the first mysterious term $\dim H^1_{\mathscr{L}_Q^\perp}(G_F, (\operatorname{ad}\overline{r})(1))$ go away.

16.3 Special choices of Q

Lemma 16.3.1. *Fix $q \geq \dim H^1(G_{F,S\cup\{w|l\}}, (\operatorname{ad}\overline{r})(1))$. Then for each $N \in \mathbb{Z}_{>0}$, there is a set Q_N of primes of F such that*

- *if $v \in Q_N$, the rational prime below v splits in F_0, $v \notin S \cup \{w|l\}$, and v is unramified in F (for the purpose of this lemma, we will say that such v is "good").*
- *if $v \in Q_N$, then $q_v \equiv 1 \pmod{l^N}$.*
- *if $v \in Q_N$, then $\overline{r}(\mathrm{Frob}_v)$ has distinct eigenvalues.*
- *$\#Q_N = q$.*
- *$H^1_{\mathscr{L}_{Q_N}^\perp}(G_F, \operatorname{ad}\overline{r}(1)) = (0)$.*

These conditions imply that $R^\square_{\chi,Q_N}$ is topologically generated over R^{loc}_χ by $qn - n^2[F^+ : \mathbb{Q}]$ elements (which is nonnegative, as will come out of the proof).

Proof. We have

$$H^1_{\mathscr{L}_Q^\perp}(G_F, \operatorname{ad}\overline{r}(1)) = \ker\left(H^1(G_{F,S\cup\{w|l\}}, \operatorname{ad}\overline{r}(1)) \to \bigoplus_{v\in Q} H^1(G_{F_v}, \operatorname{ad}\overline{r}(1))\right).$$

It suffices to prove that for all $0 \neq [\varphi] \in H^1(G_{F,S\cup\{w|l\}}, \operatorname{ad}\overline{r}(1))$, there is v such that

- v is good,
- $q_v \equiv 1 \pmod{l^N}$,

- $\overline{r}(\mathrm{Frob}_v)$ has distinct eigenvalues, and

- $\mathrm{res}_v[\varphi] \in H^1(G_{k(v)}, \mathrm{ad}\,\overline{r}(1))$ is nonzero. Since $G_{k(v)}$ is a pro-cyclic group, the cohomology is just $\mathrm{ad}\,\overline{r}(1)/(\mathrm{Frob}_v - 1)\,\mathrm{ad}\,\overline{r}(1)$, and the (1) goes away since $q_v \equiv 1 \pmod{l^N}$, so this condition is equivalent to saying that $\varphi(\mathrm{Frob}_v) \notin ((\mathrm{ad}\,\overline{r})(\mathrm{Frob}_v) - 1)\,\mathrm{ad}\,\overline{r}$.

(Then we can iteratively choose a generator of $H^1(G_{F,S\cup\{w|l\}}, \mathrm{ad}\,\overline{r}(1))$, find a v satisfying the above conditions, put it into Q_N, leaving a smaller global H^1 with at least one fewer generator, and repeat. Since we chose q large enough, we can keep going until we've eliminated every generator and made the kernel 0.) For this, it suffices by Chebotarev to prove that for all $0 \neq [\varphi] \in H^1(G_{F,S\cup\{w|l\}}, \mathrm{ad}\,\overline{r}(1))$, there is $\sigma \in G_{F(\zeta_{l^N})}$ such that

- $\overline{r}(\sigma)$ has distinct eigenvalues, and

- $\varphi(\sigma) \notin (\sigma - 1)\,\mathrm{ad}\,\overline{r}$.

(Then Chebotarev would tell us that we can find v such that Frob_v is this σ; $\sigma \in G_{F(\zeta_{l^N})}$ enforces the $q_v \equiv 1 \pmod{l^N}$ condition; goodness eliminates finitely many primes so doesn't affect Chebotarev.)

Suppose we chose a random σ and $\tau \in G_{\overline{F}^{\ker \mathrm{ad}\,\overline{r}}(\zeta_{l^N})}$. Then since $\mathrm{ad}\,\overline{r}(\tau)$ is trivial, we have $\mathrm{ad}\,\overline{r}(\tau\sigma) = \mathrm{ad}\,\overline{r}(\sigma)$, so $\overline{r}(\tau)$ still has distinct eigenvalues, and also $(\sigma-1)\,\mathrm{ad}\,\overline{r} = (\tau\sigma - 1)\,\mathrm{ad}\,\overline{r}$, so if σ falls in $(\sigma - 1)\,\mathrm{ad}\,\overline{r}$, as long as $\varphi(\tau\sigma) = \varphi(\tau) + \varphi(\sigma)$ does not fall in $(\sigma - 1)\,\mathrm{ad}\,\overline{r}$, we can choose $\tau\sigma$ instead. So it suffices to prove that for all $0 \neq [\varphi] \in H^1(G_{F,S\cup\{w|l\}}, (\mathrm{ad}\,\overline{r})(1))$, there exists $\sigma \in G_{F(\zeta_N)}$ such that

$$\varphi G_{\overline{F}^{\ker \mathrm{ad}\,\overline{r}}(\zeta_{l^N})} \not\subset (\sigma - 1)\,\mathrm{ad}\,\overline{r}$$

and $\overline{r}(\sigma)$ has n distinct eigenvalues. This would be implied if we knew that

$$\varphi|_{G_{\overline{F}^{\ker \mathrm{ad}\,\overline{r}}(\zeta_{l^N})}} \neq 1$$

because in this case the image $\varphi G_{\overline{F}^{\ker \mathrm{ad}\,\overline{r}}(\zeta_{l^N})}$ is nonzero and invariant under G_F, so spans some subrepresentation $W \subset \mathrm{ad}\,\overline{r}$, so we need σ with n distinct eigenvalues such that $\mathrm{ad}\,\overline{r}(\sigma)$ has an eigenvalue 1 on W, but this is just the enormous condition. So using the inflation-restriction sequence, it is enough to show that

$$H^1\left(\mathrm{Gal}\left(\overline{F}^{\ker \mathrm{ad}\,\overline{r}}(\zeta_{l^N})/F\right), \mathrm{ad}\,\overline{r}(1)\right) \neq (0).$$

This is good because finite Galois groups are less mysterious than absolute Galois groups. We claim that $F(\zeta_{l^n})$ and $\overline{F}^{\ker \mathrm{ad}\,\overline{r}}(\zeta_l)$ are linearly disjoint over $F(\zeta_l)$. This is because the former has l-power order over $F(\zeta_l)$ and the latter has no quotient of order l because of the enormous condition. Thus let

$$H = \mathrm{Gal}\left(\overline{F}^{\ker \mathrm{ad}\,\overline{r}}(\zeta_l)/F(\zeta_l)\right) = \mathrm{Gal}\left(\overline{F}^{\ker \mathrm{ad}\,\overline{r}}(\zeta_{l^N})/F(\zeta_{l^n})\right).$$

We know that

$$H^1\left(\mathrm{Gal}(F(\zeta_l)/F), (\mathrm{ad}\,\overline{r})(1)^{\mathrm{Gal}(\overline{F}/F(\zeta_l))}\right) = (0)$$

because $\mathrm{Gal}(F(\zeta_l)/F)$ has order prime to l (dividing $l-1$) and $(\mathrm{ad}\,\overline{r})(1)^{\mathrm{Gal}(\overline{F}/F(\zeta_l))}$ is a $\mathbb{F}$-vector space. So, again by inflation-restriction, what we want to know is that

$$H^1\left(\mathrm{Gal}\left(\overline{F}^{\ker \mathrm{ad}\,\overline{r}}(\zeta_{l^N})/F(\zeta_l)\right), \mathrm{ad}\,\overline{r}(1)^{\mathrm{Gal}(F(\zeta_l)/F)}\right) = (0).$$

Again by inflation-restriction, we can split this up and first look at

$$H^1\left(\mathrm{Gal}\left(\overline{F}^{\ker \mathrm{ad}\,\overline{r}}(\zeta_{l^N})/F(\zeta_l)\right), \mathrm{ad}\,\overline{r}^H\right)(1)^{\mathrm{Gal}(F(\zeta_l)/F)}$$

(we pulled the (1) out because H doesn't act on it); by irreducibility, $\mathrm{ad}\,\overline{r}^H = \mathbb{F}$, so this is

$$= \mathrm{Hom}(\mathrm{Gal}(F(\zeta_{l^N})/F(\zeta_l)), \mathbb{F})(1)^{\mathrm{Gal}(F(\zeta_l)/F)} = (0).$$

because $\mathrm{Gal}(F(\zeta_l)/F)$ acts trivially on homomorphisms out of $\mathrm{Gal}(F(\zeta_{l^N})/F(\zeta_l))$, since $F(\zeta_{l^N})/F(\zeta_l)$ is an abelian extension, and nontrivially on the twist (1). Finally we want to check that

$$H^1(H, \mathrm{ad}\,\overline{r})(1)^{\mathrm{Gal}(F(\zeta_l)/F)} = (0)$$

but in fact $H^1(H, \mathrm{ad}\,\overline{r}) = 0$ because H is enormous. □

This completes the algebraic number theory input. Next time, we'll start patching these things over varying Q to get an infinite-level picture that we can analyze.

17 May 25: patching part 1.

17.1 Patching setup

We saw last time that if we fix $q \geq \dim H^1\left(G_{F,S\cup\{w|l\}}, (\mathrm{ad}\,\overline{r})(1)\right)$ (with $S = R \cup \{v_0, v_0^c\}$), then for all $N \in \mathbb{Z}_{>0}$, we can choose Q_N such that $\#Q_N = q$, v is good for $v \in Q_N$, $q_v \equiv 1 \pmod{l^N}$, and $\overline{r}(\mathrm{Frob}_v)$ has distinct eigenvalues $\alpha_{v,1}, \ldots, \alpha_{v,n}$. We also set $\chi : \prod_{v\in R}(k(v)^\times)^n \to \mathcal{O}^\times$ of l-power order, where χ was either 1 or χ_0, where $\chi_{0,v,i} \neq \chi_{0,v,j}$ if $i \neq j$.

Given these choices, we looked at the deformation ring R_{χ,Q_N} with additional ramification allowed at primes in Q_N, which is an algebra over $\mathcal{O}[\Delta_{Q_N}]$, where Δ_{Q_N} is the maximal l-power quotient of $\prod_{v\in Q_N}(k(v)^\times)^n$; in particular note that we have $\prod_{v\in Q_N}(k(v)^\times)^n \twoheadrightarrow \left(\mathbb{Z}/l^N\mathbb{Z}\right)^{nq}$. If $\mathfrak{a}_{Q_N} \trianglelefteq \mathcal{O}[\Delta_{Q_N}]$ is the augmentation ideal, we saw that $R_{\chi,Q_N}/\mathfrak{a}_{Q_N} \xrightarrow{\sim} R_{\chi,\emptyset}$.

We also looked at the framed deformations

$$R^{\square}_{\chi,Q_N} \cong R_{\chi,Q_N}\widehat{\otimes}_{\mathcal{O}}\mathscr{T}$$

where $\mathscr{T}$ is a power series ring in $n^2\#(S\cup\{w|l\}) - 1$ variables (this expression depends on choices, but we made these choices consistently). So $R^{\square}_{\chi,Q_N}$ is an algebra over $\mathscr{T}[\Delta_{Q_N}]$, and $\mathscr{T}[\Delta_{Q_N}]$ has an ideal

$$\tilde{\mathfrak{a}}_{Q_N} := \mathfrak{a}_{Q_N} + \mathfrak{a}_{\mathscr{T}}$$

where $\mathfrak{a}_{\mathscr{T}} = \ker(\mathscr{T} \to \mathcal{O})$ for the map $\mathscr{T} \to \mathcal{O}$ sending all variables to 0. Then we also get an isomorphism

$$R^{\square}_{\chi,Q_N}/\tilde{\mathfrak{a}}_{Q_N} \xrightarrow{\sim} R_{\chi,\emptyset}.$$

All of these objects are independent of $\chi \pmod{\lambda}$.

Let

$$S_\infty := \mathscr{T}[\![\mathbb{Z}_l^{nq}]\!] \cong \mathcal{O}[\![T_1, \ldots, T_{n^2\#(S\cup\{w|l\})-1+nq}]\!] \twoheadrightarrow \mathscr{T}[\Delta_{Q_N}]$$

since adjoining the group $\mathbb{Z}_l$ is like adjoining a topological generator of it minus 1, and we have a surjection $\mathbb{Z}_l^{nq} \twoheadrightarrow \Delta_{Q_N}$ (with kernel contained in $l^N\mathbb{Z}_l^{nq}$). Let

$$\mathfrak{a}_\infty := \langle \mathfrak{a}_{\mathscr{T}}, \gamma - 1 \mid \gamma \in \mathbb{Z}_l^{nq} \rangle \trianglelefteq S_\infty$$

so we have $\mathfrak{a}_\infty \twoheadrightarrow \tilde{\mathfrak{a}}_{Q_N}$. The advantage of S_∞ is that it no longer depends on N; it is the limit of the rings $\mathscr{T}[\Delta_{Q_N}]$ as $N \to \infty$, though not in a canonical way (as in the maps between the rings are not canonical, but we can choose them as desired). This gives a map $S_\infty \to R^\square_{\chi,Q_N}$ such that

$$R^\square_{\chi,Q_N}/\mathfrak{a}_\infty = R_{\chi,\emptyset}.$$

We also have a surjection

$$R^\square_{\chi,\infty} := R^{loc}_\chi[\![x_1, \ldots, x_{nq-[F^+:\mathbb{Q}]n^2}]\!] \twoheadrightarrow R^\square_{\chi,Q_N}$$

which again depends on choices, but we can choose these surjections to be compatible mod λ as χ varies (first choose a surjection mod λ, which works simultaneously for all χ, then lift it for each χ). We are going to see that as N gets larger, $R^\square_{\chi,Q_N}$ looks more and more like $R^\square_{\chi,\infty}$, which involves no choice of Q_N. We have

$$\dim R^\square_{\chi,\infty} = n^2\#(S \cup \{w|l\}) + \frac{n(n-1)}{2}[F : \mathbb{Q}] + 1 + nq - [F^+ : \mathbb{Q}]n^2$$

(the $+1$ is just for the Krull dimension of $\mathcal{O}$). This can be rewritten

$$\dim R^\square_{\chi,\infty} = \dim S_\infty + 1 - n[F^+ : \mathbb{Q}].$$

Notice that the difference $n[F^+ : \mathbb{Q}] - 1$ is exactly the length of the range where the cohomology of the locally symmetric space doesn't vanish. Now we need to patch the rings $R^\square_{\chi,Q_N}$ together as N varies in order to make sense of the claim that they approach $R^\square_{\chi,\infty}$. In fact there is no natural relationship between the different $R^\square_{\chi,Q_N}$, hence no natural maps, so we need to force out unnatural maps by a compactness argument. That requires working with finite objects. These are power series rings, not finite rings, but they're the inverse limits of their finite quotients, which we now have to replace them by.

Let $J \trianglelefteq S_\infty$ be an open ideal and $d \in \mathbb{Z}_{>0}$. Consider the complex

$$\mathscr{C}(J,N)_\chi := \mathscr{C}_{\chi,Q_N} \otimes^L_{\mathcal{O}[\Delta_{Q_N}]} S_\infty/J \in D^b(S_\infty/J)$$

which makes sense for all but finitely many N, because we have a map $\mathcal{O}[\Delta_{Q_N}] \to S_\infty/J$ as long as

$$\ker(\mathcal{O}[\![\mathbb{Z}_l^{nq}]\!] \twoheadrightarrow \mathcal{O}[\Delta_{Q_N}]) \subset J$$

which is true for almost all N because these kernels shrink and are cofinal and J is open. Note that

$$\mathscr{C}(J,N)_1 \otimes^L_{S_\infty/J} S_\infty/(J,\lambda) \cong \mathscr{C}(J,N)_{\chi_0} \otimes^L_{S_\infty/J} S_\infty/(J,\lambda)$$
$$\mathscr{C}(J,N)_\chi \otimes^L_{S_\infty/J} S_\infty/(\mathfrak{a}_\infty, J) \cong \mathscr{C}_{\chi,\emptyset} \otimes^L_{\mathcal{O}} \mathcal{O}/J$$

(where by $\mathcal{O}/J$ we mean we mod out $\mathcal{O}$ by the image of J—some finite power of λ—in $S_\infty/\mathfrak{a}_\infty \cong \mathcal{O}$) and these two isomorphisms are compatible. The point is that for each J, we are listing out a countable number of candidates for what our objects can be mod J.

Let

$$\mathbb{T}(J,N)_\chi = \operatorname{im}\left(\mathbb{T}_{\chi,Q_N} \otimes_{\mathcal{O}[\Delta_{Q_N}]} S_\infty/J \to \operatorname{End}(\mathscr{C}(J,N)_\chi)\right)$$

so that we have a surjection

$$\mathbb{T}(J,N)_\chi/\mathfrak{a}_\infty \twoheadrightarrow \operatorname{im}\left(\mathbb{T}_{\chi,\emptyset} \to \operatorname{End}(\mathscr{C}_{\chi,\emptyset} \otimes^L_{\mathcal{O}} \mathcal{O}/J)\right).$$

Again these are compatible mod λ.

17.2 Minimal representatives of complexes

Lemma 17.2.1. *Suppose R is a noetherian local ring and $\mathscr{C} \in D^b(R)$ is perfect, i.e. represented by a bounded complex of finite projective R-modules (which since R is local means the same thing as finite free R-modules). Then $\mathscr{C}$ can be represented by a bounded complex $C^\bullet$ of finite projective R-modules such that $C^\bullet \otimes R/\mathfrak{m}_R$ has all 0 differentials. ($C^\bullet$ is called a minimal representative of $\mathscr{C}$.) In this case*

$$\operatorname{rank} C^i = \dim H^i(\mathscr{C} \otimes^L_R R/\mathfrak{m}).$$

(This is just because $\mathscr{C} \otimes^L_R R/\mathfrak{m}$ is represented by $C^\bullet \otimes R/\mathfrak{m}_R$, which has zero differentials, so each term is isomorphic to its cohomology, and we can calculate $\operatorname{rank} C^i$ after reducing mod $\mathfrak{m}_R$.)

If $\mathscr{D} \in D^b(R)$ is also perfect, $D^\bullet$ is a minimal representative of $\mathscr{D}$, and we have $f : \mathscr{C} \to \mathscr{D}$, then f is represented by a map of complexes $\tilde{f} : C^\bullet \to D^\bullet$ (this is a more general fact—if we represent $\mathscr{C}$ by a complex of projectives then a map f out of $\mathscr{C}$ in the derived category can be realized by a map of complexes out of that complex). If f is an isomorphism in $D^b(R)$, then $\tilde{f}$ is an isomorphism of complexes, not just a quasi-isomorphism.

Proof. Choose a perfect complex $C^\bullet$ representing $\mathscr{C}$ such that $\sum_i \operatorname{rank} C^i$ is minimal. We want to show that the differentials in $C^\bullet \otimes_R R/\mathfrak{m}_R$ are 0. Suppose not, and

$$d : C^i \otimes_R R/\mathfrak{m}_R \to C^{i+1} \otimes R/\mathfrak{m}_R$$

is not zero. We are going to find a representing complex of lower rank.

Choose $\overline{e}_1 \in C^i \otimes R/\mathfrak{m}_R$ with $d\overline{e}_1 \neq 0$. Choose a basis $\overline{e}_1, \dots, \overline{e}_r$ of $C^i \otimes R/\mathfrak{m}_R$ and a basis $d\overline{e}_1, \overline{f}_2, \dots, \overline{f}_s$ of $C^{i+1} \otimes R/\mathfrak{m}_R$. By Nakayama's lemma, we can lift these to bases $e_1, \dots, e_r$ of C^i and $de_1, f_2, \dots, f_s$ of C^{i+1} (in particular Nakayama says that any lifts will form a basis, so we can use de_1). For $i > 1$, replace e_i by $e'_i = e_i - \alpha_i e_1$ so that $de'_i \in \langle f_2, \dots, f_s \rangle$; this is still a basis since the transformation is upper triangular. Then

$$\ker d \subset \langle e'_2, \dots, e'_r \rangle, \qquad \operatorname{im}(d) \supset \langle de_1 \rangle.$$

So replace the relevant terms of our complex with

$$\cdots C^{i-1} \to \langle e'_2, \dots, e'_r \rangle \xrightarrow{d} \langle f_2, \dots, f_s \rangle \to C^{i+2} \to \cdots.$$

This has a natural map to our original complex $C^\bullet$ which we can check is a quasi-isomorphism: write out the cohomology groups at i and $i+1$ and check that they are unchanged. In particular at i, we get $\ker d/\operatorname{im}(C^{i-1})$, but $\ker d$ on all of C^i is contained in $\langle e_2', \ldots, e_r' \rangle$, so $\operatorname{im}(C^{i-1})$ is also contained in $\langle e_2', \ldots, e_r' \rangle$, and the quotient doesn't change because the kernel and the image are both the same. Similarly at $i+1$, the kernel of the original map $C^{i+1} \to C^{i+2}$ is the kernel on $\langle f_2, \ldots, f_s \rangle$ plus de_1, and the image of the original d is the image of $\langle f_2, \ldots, f_s \rangle$ plus de_1, so again the cohomology group is unchanged. This is our desired contradiction.

For the final assertion, we know that

$$C^i \otimes R/\mathfrak{m}_R \xrightarrow{\sim} H^i(\mathscr{C} \otimes^L_R R/\mathfrak{m}_R)$$

$$D^i \otimes R/\mathfrak{m}_R \xrightarrow{\sim} H^i(\mathscr{D} \otimes^L_R R/\mathfrak{m}_R)$$

and we are given that $H^i(f)$ is an isomorphism between the targets, so $\tilde{f} : C^i \otimes R/\mathfrak{m}_R \xrightarrow{\sim} D^i \otimes R/\mathfrak{m}_R$ is an isomorphism, so $\det(\tilde{f} \pmod{\mathfrak{m}_R}) \neq 0$, do $\det(\tilde{f})$ is a unit, so $\tilde{f}$ is an isomorphism. □

17.3 Application to our situation

$\mathscr{C}(J,N)_\chi$ is perfect, so we can choose a minimal representative $C^\bullet(J,N)_\chi$. Furthermore

$$\operatorname{rank}_{S_\infty/J} C^i(J,N)\chi = \dim H^i(\mathscr{C}_{1,\emptyset} \otimes^L_{\mathcal{O}} \mathbb{F})$$

which is independent of J, N, and χ. (This is important because we wouldn't be able to glue these things together if their ranks kept getting bigger!)

Note that $\mathbb{T}(J,N)_\chi$ is artinian local of length bounded independently of N and χ. This is because it suffices to say the same for the preimage of $\mathbb{T}(J,N)_\chi$ in $\operatorname{End}_{S_\infty/J}(C^\bullet(J,N)_\chi)$ (every endomorphism of $\mathscr{C}(J,N)_\chi$ comes from an endomorphism of $C^\bullet(J,N)_\chi$). But that endomorphism ring has finite length over S_∞ bounded independently of N and χ, because $C^\bullet(J,N)_\chi$ is finite free over S_∞/J and thus bounded. So now everything on the Hecke algebra side has finite cardinality.

Now for the same thing on Galois deformation rings: we have

$$R_{\chi,Q_N} \twoheadrightarrow \mathbb{T}_{\chi,Q_N}/I_{\chi,Q_N}$$

$$R^\square_{\chi,Q_N} \widehat{\otimes}_{\mathscr{T}[\Delta_{Q_N}]} S_\infty/J = R_{\chi,Q_N} \widehat{\otimes}_{\mathcal{O}[\Delta_{Q_N}]} S_\infty/J \twoheadrightarrow \mathbb{T}(J,N)_\chi/I(J,N)_\chi$$

where $I(J,N)^\delta_\chi = (0)$ for δ independent of J, N, χ. These are not of finite cardinality. So let

$$R(d,J,N)_\chi = R_{\chi,Q_N}/\mathfrak{m}^d \otimes_{\mathcal{O}[\Delta_{Q_N}]} S_\infty/J.$$

Then we have $R_{\infty,\chi} \twoheadrightarrow R(d,J,N)_\chi$, $S_\infty \to R(d,J,N)_\chi$, and

$$\mathfrak{m}^{e(J,d)}_{R_{\infty,\chi}} \to (0) \subset R(d,J,N)_\chi$$

for some $e(J,d)$ only depending on J and d, because we have $\mathfrak{m}^{e(J)}_{S_\infty} \subset J$ for some $e(J)$, and since $\mathfrak{m}_{R_{\infty,\chi}}$ is the sum of the maximal ideals from R_{χ,Q_N} and S_∞, $e(J,d) = d + e(J)$ works.

We have

$$R(d,J,N)_\chi \twoheadrightarrow \mathbb{T}(J,N)_\chi / I(J,N)_\chi$$

if $d \gg_J 0$, independently of N and χ, since $\mathbb{T}(J,N)_\chi/I(J,N)_\chi$ has finite length bounded independently of N and χ. These are again compatible mod λ. Finally, if $S_\infty \trianglerighteq J_1 \supset J_2$, then

$$\begin{aligned}\mathscr{C}(J_1,N)_\chi \otimes^L_{S_\infty/J_1} S_\infty/J_2 &\cong \mathscr{C}(J_2,N)_\chi \\ \mathbb{T}(J_1,N)_\chi &\twoheadrightarrow \mathbb{T}(J_2,N)_\chi \\ I(J_1,N)_\chi &\twoheadrightarrow I(J_2,N)_\chi.\end{aligned}$$

Now we have infinitely many N, and for each d and J we have finitely many possible choices of $R(d,J,N)_\chi$, $\mathbb{T}(J,N)_\chi/I(J,N)_\chi$, and the maps between them, so we can pigeonhole. For each d and J, we choose a set of such data that occurs for infinitely many N, then go to the next d and J and choose a compatible set that still occurs for infinitely many N, and so on. Then we can patch those things together and take a limit. We are going to do this with ultrafilters because that is the trend these days, though it's just the same argument anyway.

17.4 Ultrafilters

Definition 17.4.1. A non-principal ultrafilter $\mathscr{F}$ on $\mathbb{Z}_{>0}$ is a collection of subsets $\mathbb{Z}_{>0}$ such that

- $\mathbb{Z}_{>0} \in \mathscr{F}$, $\emptyset \notin \mathscr{F}$.
- if $F_1, F_2 \in \mathscr{F}$ then $F_1 \cap F_2 \in \mathscr{F}$.
- if $F_1 \in \mathscr{F}$ and $F_2 \supset F_1$, then $F_2 \in \mathscr{F}$. (These so far are the conditions to be a filter.)
- if $F \subset \mathbb{Z}_{>0}$ then either F or $\mathbb{Z}_{>0} \setminus F$ is in $\mathscr{F}$. (This makes it an ultrafilter.)
- there is no finite set in $\mathscr{F}$. (Having a finite set would force $\mathscr{F}$ to be the set of everything containing a particular integer, so this condition makes it non-principal).

Proof that ultrafilters exist. Use Zorn's lemma to choose a maximal filter $\mathscr{F}$ containing no finite set. Suppose $F, \mathbb{Z}_{>0} \setminus F \notin \mathscr{F}$. Then consider the two obvious strictly bigger filters:

$$\begin{aligned}\mathscr{F}' &= \{H \mid H \supset F \cap G \text{ where } G \in \mathscr{F}\}, \\ \mathscr{F}'' &= \{H \mid H \supset (\mathbb{Z}_{>0} \setminus F) \cap G \text{ where } G \in \mathscr{F}\}\end{aligned}$$

(so we're either adding in F or $\mathbb{Z}_{>0} \setminus F$, plus everything else that has to go in with them). Since $\mathscr{F}$ is maximal, both must contain a finite set: there must be $G' \in \mathscr{F}$ such that $G' \cap F$ is finite, and $G'' \in \mathscr{F}$ such that $G'' \cap (\mathbb{Z}_{>0} \setminus F)$ is finite. But then $G' \cap G'' \in \mathscr{F}$, and

$$G' \cap G'' = (G' \cap G'' \cap F) \cup (G' \cap G'' \cap (\mathbb{Z}_{>0} \setminus F))$$

is finite, a contradiction since $\mathscr{F}$ contains no finite set. We conclude that $\mathscr{F}$ is an ultrafilter. ∎

Now let A be a local ring of finite cardinality. Then $\prod_{\mathbb{Z}_{>0}} A$ contains the prime ideal

$$\mathfrak{p}_{\mathscr{F}} := \left\{(a_N) \in \prod A \mid F_{(a_N)} := \{N \mid a_N \in \mathfrak{m}_A\} \in \mathscr{F}\right\}.$$

This is an ideal because $F_{(a_N)}$ gets strictly bigger when you multiply (a_N) by something in $\prod A$ and filters are closed under supersets. It is prime because $\mathfrak{m}_A$ is prime, so $F_{(a_N)(b_N)} = F_{(a_N)} \cup F_{(b_N)}$, so if $(a_N)(b_N) \in \mathfrak{p}_{\mathscr{F}}$ then $F_{(a_N)} \cup F_{(b_N)} \in \mathscr{F}$ so either $F_{(a_N)} \in \mathscr{F}$ or $F_{(b_N)} \in \mathscr{F}$.

Lemma 17.4.2. *Suppose $(a_N) \in \prod_{\mathbb{Z}_{>0}} A$. Then $[(a_N)] \in (\prod A)_{\mathfrak{p}_F}$ is 0 if and only if $\{N : a_N = 0\} \in \mathscr{F}$.*

Proof. $(a_N) \mapsto 0 \in (\prod A)_{\mathfrak{p}_F}$ if and only if there is $(b_N) \notin \mathfrak{p}_{\mathscr{F}}$ such that $b_N a_N = 0$ for all N. But $(b_N) \notin \mathfrak{p}_{\mathscr{F}}$ is equivalent to $\{N \mid b_N \in A^\times\} \in \mathscr{F}$. So we can choose (b_N) to be a unit at those indices and 0 everywhere else, and conclude that this is equivalent to $\{N \mid a_N = 0\} \in \mathscr{F}$. ☐

Corollary 17.4.3. *We have $A \xrightarrow{\sim} \left(\prod_{\mathbb{Z}_{>0}} A\right)_{\mathfrak{p}_{\mathscr{F}}}$.*

So we get the same ring back again, but if we had a bunch of equations, one for each copy, then they hold in the localization if and only if the set of indices for which they hold in the product is in the filter—they don't have to hold everywhere.

18 May 27: patching part 2.

Last time we started trying to patch our auxiliary deformation rings and Hecke algebras associated to sets of primes Q_N to get an infinite version. We will now continue with this.

18.1 More ultrafilters

Let $\mathscr{F}$ be a non-principal ultrafilter on $\mathbb{Z}_{>0}$, i.e. a collection of subsets of $\mathbb{Z}_{>0}$ which is closed under finite intersections and supersets, with no finite sets, and which for every set contains either it or its complement ("every time you divide $\mathbb{Z}_{>0}$ into a finite number of subsets, it chooses a preferred one of those subsets").

Let A be a local ring with $\#A < \infty$. Inside $\prod_{\mathbb{Z}_{>0}} A$ we have a prime ideal

$$\mathfrak{p}_{\mathscr{F}} = \left\{(a_N) \in A^{\mathbb{Z}_{>0}} \mid F_{(a_N)} \in \mathscr{F}\right\}$$

where $F_{(a_N)} = \{N \mid a_N \in \mathfrak{m}_A\}$.

Lemma 18.1.1. *Suppose M_N is an A-module for all N. Then*

$$\ker\left(\prod_{\mathbb{Z}_{>0}} M_N \to \left(\prod M_N\right)_{\mathfrak{p}_{\mathscr{F}}}\right) = \{(M_N) \mid \{N \mid M_N = 0\} \in \mathscr{F}\}$$

and

$$\prod_{\mathbb{Z}_{>0}} M_N \twoheadrightarrow \left(\prod M_N\right)_{\mathfrak{p}_{\mathscr{F}}}$$

is surjective.

Proof. $(M_N) \mapsto 0$ if and only if there is $(b_N) \in (\prod A) \setminus \mathfrak{p}_{\mathscr{F}}$ such that $b_N M_N = 0$ for all N; this is true if and only if there is $(b_N) \in \prod A$ such that $\{N \mid b_N \in A^\times\} \in \mathscr{F}$ and $b_N M_N = 0$ for all N; this implies that $\{N \mid M_N = 0\}$ contains $\{N \mid b_N \in A^\times\}$, hence is also in $\mathscr{F}$, and conversely if $\{N \mid M_N = 0\} \in \mathscr{F}$ then we can choose b_N to be 1 at $\{N \mid M_N = 0\}$ and 0 everywhere else.

For surjectivity, take any element of the localization, say $(M_N)/(a_N)$ with $(a_N) \notin \mathfrak{p}_{\mathscr{F}}$, i.e. $\{N \mid a_N \in A^\times\} \in \mathscr{F}$. Let

$$M'_N = \begin{cases} a_N^{-1} M_N & \text{if } a_N \in A^\times \\ 0 & \text{otherwise.} \end{cases}$$

Then $(a_N)(M'_N) - (M_N) \mapsto 0$ in $(\prod M_N)_{\mathfrak{p}_{\mathscr{F}}}$, since this is true locally at $\{N \mid a_N \in A^\times\} \in \mathscr{F}$ by construction, therefore true in the localization by Part 1. Hence $(M'_N) \mapsto (M_N)/(a_N)$. □

Lemma 18.1.2. *1. If M is a finitely generated A-module and $M \twoheadrightarrow M_N$ for all N, then $M \twoheadrightarrow (\prod M_N)_{\mathfrak{p}_{\mathscr{F}}}$.*

2. If M is a finitely generated A-module, then $M \xrightarrow{\sim} \left(\prod_{\mathbb{Z}_{>0}} M\right)_{\mathfrak{p}_{\mathscr{F}}}$.

Proof. Since M is finitely generated over A and $\#A < \infty$, also $\#M < \infty$.

1. Let $(m_N) \in \prod M_N$. If $m \in M$, let $F_m = \{N \mid m \mapsto m_N\}$. We have $\mathbb{Z}_{>0} = \bigcup_{m \in M} F_m$ since $M \twoheadrightarrow M_N$ for all N. This is a finite union. So $F_m \in \mathscr{F}$ for some m, and for this m we have $m \mapsto (m_N) \in (\prod M_N)_{\mathfrak{p}_{\mathscr{F}}}$.

2. We have $M \twoheadrightarrow (\prod M)_{\mathfrak{p}_{\mathscr{F}}}$. Suppose $m \mapsto 0$. Then $\{N \mid m = 0\} \in \mathscr{F}$. This is $\emptyset$ if $m \neq 0$ and $\mathbb{Z}_{>0}$ if $m = 0$; since $\emptyset$ is not in $\mathscr{F}$ and $\mathbb{Z}_{>0}$ is, we conclude that $m = 0$.

□

Corollary 18.1.3. $A \xrightarrow{\sim} \left(\prod_{\mathbb{Z}_{>0}} A\right)_{\mathfrak{p}_{\mathscr{F}}}$.

So if M_N is an A-module for each N, so is $\left(\prod_{\mathbb{Z}_{>0}} M_N\right)_{\mathfrak{p}_{\mathscr{F}}}$.

Lemma 18.1.4. *If M_N, M'_N are two collections of A-modules and $\{N \mid M_N \cong M'_N\} \in \mathscr{F}$, then*

$$\left(\prod M_N\right)_{\mathfrak{p}_{\mathscr{F}}} \cong \left(\prod M'_N\right)_{\mathfrak{p}_{\mathscr{F}}}$$

Proof. The idea is to construct maps in both directions using the given isomorphisms and 0. That is, for the map from the LHS to the RHS, use an isomorphism at $\{N \mid M_N \cong M'_N\}$ and 0 everywhere else; for the map from the RHS to the LHS, do the same with the local inverse isomorphisms. The composite of these two maps in either order is the identity at $\{N \mid M_N \cong M'_N\} \in \mathscr{F}$, and the previous lemma implies that two endomorphisms of either module are equal if equal at a set in $\mathscr{F}$, so the constructed maps are inverse isomorphisms. □

Lemma 18.1.5. *If* $0 \to M_N \to M'_N \to M''_N \to 0$ *is exact for all* N*, then*

$$0 \to \left(\prod M_N\right)_{\mathfrak{p}_{\mathscr{F}}} \to \left(\prod M'_N\right)_{\mathfrak{p}_{\mathscr{F}}} \to \left(\prod M''_N\right)_{\mathfrak{p}_{\mathscr{F}}} \to 0$$

is exact.

Corollary 18.1.6. *If* M_N *is finitely generated over* A *for all* N *and the number of generators is bounded independently of* N*, then* $(\prod M_N)_{\mathfrak{p}_{\mathscr{F}}}$ *is finitely generated over* A*.*

Proof. There is d such that $A^d \twoheadrightarrow M_N$ for all N, so

$$A^d = \left(\prod A\right)^d_{\mathfrak{p}_{\mathscr{F}}} = \left(\prod A^d\right)_{\mathfrak{p}_{\mathscr{F}}} \twoheadrightarrow \left(\prod M_N\right)_{\mathfrak{p}_{\mathscr{F}}}.$$

□

Lemma 18.1.7. *If* P_N *is projective over* A *for all* N *and* $\operatorname{rank}_A P_N$ *is bounded independently of* N*, then* $(\prod P_N)_{\mathfrak{p}_{\mathscr{F}}}$ *is finitely generated and projective.*

Proof. Let $F_s = \{N \mid \operatorname{rank}_A P_N = s\}$. Then $F_s = \emptyset$ if $s \gg 0$ and $\coprod_s F_s = \mathbb{Z}_{>0}$, so there is some $F_s \in \mathscr{F}$. Then

$$\left(\prod P_N\right)_{\mathfrak{p}_{\mathscr{F}}} \cong \left(\prod A^s\right)_{\mathfrak{p}_{\mathscr{F}}} \cong A^s$$

(because the first isomorphism is true locally at $F_s \in \mathscr{F}$). □

Lemma 18.1.8. *Suppose that* $\mathscr{C}_N \in D^b(A)$ *is perfect for all* N *and that* $\dim H^i(\mathscr{C}_N \otimes^L_A A/\mathfrak{m})$ *is strictly bounded independently of* N*, meaning that for all* i *it is bounded independently of* N *and also there is* i_0, i_1 *such that* $H^i(\mathscr{C}_N \otimes^L_A A/\mathfrak{m}) = (0)$ *for all* N *if* $i > i_1$ *or* $i < i_0$*. If* $C^\bullet_N$ *is a minimal representative of* $\mathscr{C}_N$*, then*

$$\left(\prod C^\bullet_N\right)_{\mathfrak{p}_{\mathscr{F}}}$$

represents a perfect element of $D^b(A)$ *which is canonically independent of the choice of* $\mathscr{C}^\bullet_N$*. We will denote this complex by*

$$\left(\prod \mathscr{C}^\bullet_N\right)_{\mathfrak{p}_{\mathscr{F}}}.$$

(This just follows from the properties of minimal representatives we stated—they have canonical isomorphisms as complexes between them.)

Lemma 18.1.9. *If* $J \trianglelefteq A$ *and* M_N *are* A*-modules, then*

$$\left(\prod M_N\right)_{\mathfrak{p}_{\mathscr{F}}} \otimes_A A/J \cong \left(\prod (M_N \otimes_A A/J)\right)_{\mathfrak{p}_{\mathscr{F}}}.$$

Proof. Since A has finite cardinality, it is artinian, hence also noetherian, hence J is finitely generated over A. So $\prod_{\mathbb{Z}_{>0}} J$ is finitely generated over $\prod_{\mathbb{Z}_{>0}} A$ (with the generators being products of copies of a generator of J over A). So $\prod_{\mathbb{Z}_{>0}} A/J$ is finitely *presented* over $\prod_{\mathbb{Z}_{>0}} A$. Therefore

$$\prod A/J \otimes_{\prod A} \prod M_N \cong \prod \left(\left(\prod A/J\right) \otimes_{\prod A} M_N\right)$$

(i.e. assuming finite presentation you can interchange the direct and tensor products—see the Stacks Project) and the RHS can be rewritten as

$$\cong \prod \left(\left(\prod A/J \right) \otimes_{\prod A} A \otimes_A M_N \right)$$

with the copy of A in the nth place. Since $(\prod A/J) \otimes_{\prod A} A \cong A/J$ (A/J has the correct universal property), we are done. □

Corollary 18.1.10. *Suppose $\mathscr{C}_N \in D^b(A)$ is perfect and $\dim H^i(\mathscr{C}_N \otimes_L A/\mathfrak{m}_A)$ is strictly bounded, and $I \trianglelefteq A$. Then*

$$\left(\prod \mathscr{C}_N \right)_{\mathfrak{p}_{\mathscr{F}}} \otimes_A^L A/I \cong \left(\prod (\mathscr{C}_N \otimes_A^L A/I) \right)_{\mathfrak{p}_{\mathscr{F}}}.$$

This is because the claim is true on the level of complexes upon taking a minimal representative.

18.2 Application to our situation

We constructed $\mathscr{C}(J,N)_\chi$ for $N \gg_J 0$. We need to compare these as J varies, but we can't do that for any fixed N since then they are only well-defined for finitely many J. Instead consider

$$\mathscr{C}(J,\infty)_\chi := \left(\prod_N \mathscr{C}(J,N)_\chi \right)_{\mathfrak{p}_{\mathscr{F}}} \in D^b(S_\infty/J)$$

where we put any random thing for $\mathscr{C}(J,N)_\chi$ at the finite number of places where it is not well-defined. Then

$$C^\bullet(J,\infty)_\chi := \left(\prod_N C^\bullet(J,N)_\chi \right)_{\mathfrak{p}_{\mathscr{F}}}$$

is a minimal representative of $\mathscr{C}(J,\infty)_\chi$. Let

$$\mathbb{T}(J,\infty)_\chi := \operatorname{im}\left(\left(\prod_N \mathbb{T}(J,N)_\chi \right)_{\mathfrak{p}_{\mathscr{F}}} \to \operatorname{End}(\mathscr{C}(J,\infty)_\chi) \right)$$

and

$$I(J,\infty)_\chi := \operatorname{im}\left(\left(\prod_N I(J,N)_\chi \right)_{\mathfrak{p}_{\mathscr{F}}} \to \mathbb{T}(J,\infty)_\chi \right).$$

Finally, let

$$R(d,J,\infty)_\chi = \left(\prod_N R(d,J,N)_\chi \right)_{\mathfrak{p}_{\mathscr{F}}}.$$

Remark 3. The following properties follow from what we've seen for individual N.

1. $R_{\infty,\chi}/\mathfrak{m}^{e(J,d)} \twoheadrightarrow R(d,J,\infty)_\chi$.

2. $\mathbb{T}(J,\infty)_\chi$ is finitely generated as a module over S_∞/J. $I(J,\infty)_\chi^\delta = 0$ for the same δ as before.

3. $R(d,J,\infty)_\chi \twoheadrightarrow \mathbb{T}(J,\infty)_\chi/I(J,\infty)_\chi$ for $d \gg_J 0$.

4. $\mathscr{C}(J,\infty)_\chi$ is perfect with minimal representative $C^\bullet(J,\infty)_\chi$.

5. If $J_1 \subset J_2 \trianglelefteq S_\infty$, then

$$S_\infty/J_2 \otimes^L_{S_\infty/J_1} \mathscr{C}(J_1,\infty)_\chi \cong \mathscr{C}(J_2,\infty)_\chi$$

   and in fact (maybe less canonically)

$$S_\infty/J_2 \otimes^L_{S_\infty/J_1} C^\bullet(J_1,\infty)_\chi \cong C^\bullet(J_2,\infty)_\chi.$$

6. $\mathscr{C}(J,\infty)_\chi \otimes^L_{S_\infty/J} \mathcal{O}/J \cong C_{\chi,\emptyset} \otimes^L_{\mathcal{O}} \mathcal{O}/J$, using the natural map $S_\infty \to \mathcal{O}$ on the LHS. (WLOG, using (5), we can take $J \supset \mathfrak{a}_\infty$, so that the LHS is just $\mathscr{C}(J,\infty)_\chi$.)

7. If $J_1 \subset J_2 \trianglelefteq S_\infty$, then $\mathbb{T}(J_1,\infty) \twoheadrightarrow \mathbb{T}(J_2,\infty)$ and $I(J_1,\infty) \twoheadrightarrow I(J_2,\infty)$. This is compatible with (5).

8. Everything is compatible with reduction mod λ, meaning that if you construct these things for two different χs and reduce them mod λ, you get canonical identifications between the objects for the different χs compatible with these maps.

Now we can take an inverse limit over all J, instead of being stuck with a limited number of J for each fixed N as we were before the "averaging" process. Let

$$C^\bullet_{\chi,\infty} = \varprojlim_r C^\bullet(\mathfrak{m}^r_{S_\infty},\infty)_\chi.$$

This represents some $\mathscr{C}_{\chi,\infty} \in D^b(S_\infty)$. Similarly, let

$$I_{\chi,\infty} = \varprojlim I(\mathfrak{m}^r_{S_\infty},\infty)_\chi \trianglelefteq \mathbb{T}_{\chi,\infty} = \varprojlim \mathbb{T}(\mathfrak{m}^r_{S_\infty},\infty)_\chi.$$

The following properties are preserved after these limits.

1. $\mathscr{C}_{\chi,\infty}$ is perfect and $C^\bullet_{\chi,\infty}$ is a minimal representative. We have

$$C^\bullet_{\chi,\infty} \otimes_{S_\infty} S_\infty/J \xrightarrow{\sim} C^\bullet(J,\infty)_\chi$$
$$\mathscr{C}_{\chi,\infty} \otimes^L_{S_\infty} \mathcal{O} \xrightarrow{\sim} \mathscr{C}_{\chi,\emptyset}.$$

   That is, we started with $\mathscr{C}_{\chi,\emptyset}$, thickened it at auxiliary primes to lie over certain group rings, and after patching now it lies over the formal power series ring S_∞.

2. $\mathbb{T}_{\chi,\infty} \hookrightarrow \mathrm{End}_{D^b(S_\infty)}(\mathscr{C}_{\chi,\infty})$. This is a Mittag-Leffler argument. We have an exact sequence

$$0 \leftarrow \mathrm{End}_{D^b(S_\infty/J)}(\mathscr{C}(J,\infty)_\chi) \leftarrow \mathrm{Hom}_{\text{complexes of } S_\infty/J\text{-modules}}(C^\bullet(J,\infty)_\chi, C^\bullet(J,\infty)_\chi)$$

Let $K_{1,J}$ be the kernel of the map $\mathrm{Hom}_{\text{complexes}} \to \mathrm{End}_{D^b(S_\infty/J)}$. Two elements of $\mathrm{Hom}_{\text{complexes}}$ have the same image in $\mathrm{End}_{D^b(S_\infty/J)}$ if they differ by a homotopy, i.e. if their difference is in the image of the map from

$$\mathrm{Hom}_{\text{graded } S_\infty/J\text{-modules}}\left(\bigoplus C^i(J,\infty)_\chi, \bigoplus C^{i-1}(J,\infty)_\chi\right)$$

given by

$$(g^i) \mapsto d \circ g^i + g^{i+1} \circ d.$$

Hence we get a surjection

$$\mathrm{Hom}_{\text{graded } S_\infty/J\text{-modules}}\left(\bigoplus C^i(J,\infty)_\chi, \bigoplus C^{i-1}(J,\infty)_\chi\right) \twoheadrightarrow K_{1,J}.$$

Let $K_{2,J}$ be the kernel of this surjection. Taking the inverse limit over all J preserves exactness of both these sequences ($K_{1,J} \to \mathrm{Hom}_{\text{complexes}} \to \mathrm{End}_{D^b(S_\infty/J)}$ and $K_{2,J} \to \mathrm{Hom}_{\text{graded}} \to K_{1,J}$) by the Mittag-Leffler condition that the successive images stabilize, since $\#K_{1,J}, \#K_{2,J} < \infty$. Since the inverse limits of $\mathrm{Hom}_{\text{complexes}}$ and $\mathrm{Hom}_{\text{graded}}$ can be identified with the corresponding terms for $J = 0$, we conclude that

$$\varprojlim_J \mathrm{End}_{D^b(S_\infty/J)}(\mathscr{C}(J,\infty)_\chi) = \mathrm{End}_{D^b(S_\infty)}(\mathscr{C}_{\chi,\infty}).$$

Since each $\mathbb{T}(J,\infty)_\chi$ embeds in its corresponding term on the LHS, the limit $\mathbb{T}_{\chi,\infty}$ embeds in the RHS.

3. $R_{\chi,\infty} \twoheadrightarrow \mathbb{T}_{\chi,\infty}/I_{\chi,\infty}$. This is because

$$\begin{aligned} R_{\chi,\infty} &\twoheadrightarrow \varprojlim R_{\infty,\chi}/\mathfrak{m}^{e(J,d(J))} \twoheadrightarrow \varprojlim R(d(J),J,\infty)_\chi \\ &\twoheadrightarrow \varprojlim \mathbb{T}(J,\infty)_\chi/I(J,\infty)_\chi \twoheadrightarrow \mathbb{T}_{\chi,\infty}/I_{\chi,\infty} \end{aligned}$$

where the individual terms of the second-to-last surjection are only defined for $d(J) \gg_J 0$, but that's fine—for each J we choose a sufficiently large $d(J)$, and that gives the desired maps in the limit. Also, in the map

$$R_{\chi,\infty} \twoheadrightarrow \varprojlim_{d,J} R(d,J,\infty)_\chi$$

we used, the RHS is an S_∞-algebra, so even though $R_{\chi,\infty}$ isn't naturally an S_∞-algebra, since S_∞ is a formally smooth power series ring, we can fill in some map $S_\infty \to R_{\chi,\infty}$ compatibly mod λ as χ varies by choosing an appropriate target for each generator of S_∞.

18.3 Reducing to the patched modules

We get a commutative diagram

$$\begin{CD}
R_{\chi,\infty}/\mathfrak{a}_\infty @>{\text{surj}}>> R_{\chi,\emptyset} \\
@V{\text{surj}}VV @VV{\text{surj}}V \\
\mathbb{T}_{\chi,\infty}/\mathfrak{a}_\infty @>{\text{surj}}>> \mathbb{T}_{\chi,\emptyset}.
\end{CD}$$

We claim that to prove the right downwards map has nilpotent kernel, it suffices to prove that the left downwards map does, or that $\ker(R_{\chi,\infty} \to \mathbb{T}_{\chi,\infty})$ is nilpotent. This would follow from the following lemma.

Lemma 18.3.1. $\ker(\mathbb{T}_{\chi,\infty}/\mathfrak{a}_\infty \to \mathbb{T}_{\chi,\emptyset})$ *is nilpotent.*

This is a general abstract commutative algebra fact:

Lemma 18.3.2. *Let A be a noetherian local ring, B an A-algebra, $I \trianglelefteq A$, $\mathscr{C} \in D^b(A)$ perfect, and $B \hookrightarrow \mathrm{End}_{D^b(A)}(\mathscr{C})$ (in particular B is A-finite, i.e. finitely generated as a module over A). Then*

$$\ker\left(B/IB \to \mathrm{End}_{D^b(A/I)}(\mathscr{C} \otimes^L_A A/I)\right)$$

is nilpotent.

Proof of Lemma 18.3.1 from Lemma 18.3.2. We apply Lemma 18.3.2 with $A = S_\infty$, $I = \mathfrak{a}_\infty$, $B = \mathbb{T}_{\chi,\infty}$, and $\mathscr{C} = \mathscr{C}_{\chi,\infty}$. We know that $\mathbb{T}_{\chi,\infty}/\mathfrak{a}_\infty$ surjects onto $\mathbb{T}_{\chi,\emptyset}$, which is a subring of $\mathrm{End}_{D^b(S_\infty/\mathfrak{a}_\infty)}$ because

$$\mathscr{C}_{\chi,\infty} \otimes^L_{S_\infty} S_\infty/\mathfrak{a}_\infty = \mathscr{C}_{\chi,\emptyset}.$$

So $\ker(\mathbb{T}_{\chi,\infty}/\mathfrak{a}_\infty \to \mathbb{T}_{\chi,\emptyset})$ is the same as the kernel in Lemma 18.3.2.

(In general, Hecke algebras don't descend as well as other patched modules—when you reduce the module the Hecke algebra is acting on, you don't know that the image of the Hecke algebra is the Hecke algebra mod the reducing ideal, just some quotient of it.) □

Here is a non-derived version of Lemma 18.3.2.

Lemma 18.3.3. *Let B be a noetherian ring, M a finitely generated faithful B-module, and $I \trianglelefteq B$. Then*

$$\ker(B/I \to \mathrm{End}(M/IM))$$

is nilpotent. (Sometimes people say B/I acts "nearly faithfully" on $\mathrm{End}(M/IM)$*.)*

Proof of Lemma 18.3.3. The idea is that "M is a nearly faithful B-module" is the same as "the support of M is all of $\operatorname{spec} B$". If $\mathfrak{p} \in \operatorname{spec} B$, then $\mathfrak{p} \supset \mathrm{ann}_B(M) = (0)$, so $M_\mathfrak{p} \neq (0)$. So if $\mathfrak{p} \in \operatorname{spec} B/I$, then

$$(M/IM)_\mathfrak{p} = M_\mathfrak{p}/I_\mathfrak{p}M_\mathfrak{p} \neq (0)$$

by Nakayama's lemma over $A_\mathfrak{p}$ (if we take a proper ideal of a local ring and a finitely generated module over it, the quotient by the ideal is 0 if and only if the module was already 0). Therefore $\mathfrak{p} \supset \mathrm{ann}_{B/I}(M/IM)$, so

$$\mathrm{ann}_{B/I}(M/IM) \subset \bigcap_{\mathfrak{p}\text{ prime of }B/I} \mathfrak{p}$$

which is nilpotent. □

Proof of Lemma 18.3.2. Choose a minimal representative $C^\bullet$ of $\mathscr{C}$. Suppose $b \notin IB$ and

$$b \in \ker \left(B/IB \to \mathrm{End}_{D^b(A/I)}(\mathscr{C} \otimes_A^L A/I) \right).$$

There is a representative $b^\bullet : C^\bullet \to C^\bullet$ of b such that $b^\bullet : C^\bullet/IC^\bullet \to C^\bullet/IC^\bullet$ is homotopic to 0, i.e.

$$b^i = d \circ k^i + k^{i+1} \circ d, \qquad k^i : C^i/I \to C^{i-1}/I.$$

Since C^i is projective, we can lift k^i to $\tilde{k}^i : C^i \to C^{i-1}$. Changing $b^\bullet$ by $d\tilde{k}^i + \tilde{k}^{i+1} \circ d$, WLOG $b^\bullet C^\bullet \subset IC^\bullet$. (So up to homotopy we can choose a representative which maps a minimal representative to I times the minimal representative.)

By the Artin-Rees lemma,

$$I \ker d \supset (I^m C^\bullet) \cap \ker d.$$

Therefore b^m acts as 0 on $H^\bullet(\mathscr{C})/IH^\bullet(\mathscr{C})$ (not the cohomology of $\mathscr{C} \otimes_A^L A/I$, but the actual cohomology of $\mathscr{C}$, reduced mod A/I). Let

$$\overline{B} = \mathrm{im}(B \to \mathrm{End}(H^\bullet(\mathscr{C}))).$$

By Lemma 18.3.3, since $\overline{B}$ acts faithfully on $H^\bullet(\mathscr{C})$, b^m must be nilpotent in $\overline{B}/I\overline{B}$, that is, $b^{mm'} \in I\overline{B}$ for some m'. But also $\ker(B \to \overline{B})$ is nilpotent (because an endomorphism of a complex that is 0 on cohomology is nilpotent—we proved this earlier with an inductive argument), which means $\ker(B/IB \to \overline{B}/I\overline{B})$ is nilpotent, so $b^{mm'm''} \in IB$ for some m''. □

Next time, we will prove that the post-patching kernel $\ker(R_{\chi,\infty} \to \mathbb{T}_{\chi,\infty})$ is nilpotent.

19 June 1: depth, dimension, and length.

Last time, we reduced our main theorem to proving that map from a patched deformation ring to a patched Hecke algebra had nilpotent kernel. This and next time, we will prove that using two main commutative algebra inputs. The first is as follows.

19.1 Depth and dimension

Let A be a local noetherian ring with maximal ideal $\mathfrak{m}$. Let M be a finitely generated A-module.

Definition 19.1.1. By an M-regular sequence $a_1, \ldots, a_d \in \mathfrak{m}$ we mean a sequence such that a_i is not a zero-divisor for $M/(a_1, \ldots, a_{i-1})M$.

Proposition 19.1.2. *Here are some basic facts about M-regular sequences.*

1. *All maximal M-regular sequences have the same length, which we call* $\mathrm{depth}_A(M)$*.*

2. $\mathrm{depth}_A(M) = \min\{i \mid \mathrm{Ext}_A^i(A/\mathfrak{m}, M) \neq (0)\}$*. (This is how you prove Part 1.)*

3. *(Auslander-Buchsbaum theorem) If M has a finite projective resolution, then*

$$\text{projective} - \text{dimension}(M) + \text{depth}_A(M) = \text{depth}_A(A),$$

where the projective dimension of M is the length of the shortest projective resolution.

4. *(Sometimes called Ischebeck's theorem) If M, N are finitely generated A-modules and $\text{depth}_A(M) > \dim(A/\text{ann}_A(N))$, where $\dim$ is Krull dimension (the length of the maximum chain of nested prime ideals minus one), then $Hom_A(N, M) = (0)$. (Part 2 is partly a special case of this: $A/\mathfrak{m}$ has dimension 0, so if $\text{depth}_A(M) > 0$ then $Hom_A(A/\mathfrak{m}, M) = 0$ and hence $\min\{i \mid \text{Ext}^i_A(A/\mathfrak{m}, M) \neq (0)\} > 0$.)*

For a proof, see e.g. Matsumura, Commutative ring theory [10]. The following consequences, which we will use, follow from the basic theory.

Lemma 19.1.3. *Let B be an A-algebra which is finite (i.e. as an A-module), and M a finitely generated B-module. (E.g. A is a ring of diamond operators like S_∞, B a Hecke algebra, and M a space of modular forms.) Then*

$$\text{depth}_A(M) = \min_{\mathfrak{n} \text{ maximal ideal of } B} \text{depth}_{B_\mathfrak{n}}(M_\mathfrak{n}).$$

Proof. Induct on $\text{depth}_A(M)$.

For the case $d = 0$, by Part 2 of Proposition 19.1.2, $\text{depth}_A(M) = 0$ if and only if we have an embedding $A/\mathfrak{m} \hookrightarrow M$, which we can extend linearly to get an embedding $B/\mathfrak{m} \hookrightarrow M$. A minimal B-submodule of $B/\mathfrak{m}$ must be isomorphic to $B/\mathfrak{n}$ for some prime ideal $\mathfrak{n}$ lying over $\mathfrak{m}$, which by the going-up theorem must be maximal. Since $B/\mathfrak{m}$ is finitely generated over the field $A/\mathfrak{m}$, it is finite-length and has such a minimal submodule $B/\mathfrak{n}$. We get an inclusion $B/\mathfrak{n} \hookrightarrow M$ for some $\mathfrak{n}$ maximal, hence an inclusion $B_\mathfrak{n}/\mathfrak{n} \hookrightarrow M_\mathfrak{n}$.

But also given any such inclusion, since $A/\mathfrak{m} \hookrightarrow B/\mathfrak{n}$ (the ideal lying below $\mathfrak{n}$ in A has to be a maximal ideal, so can only be $\mathfrak{m}$), we get a composite inclusion $A/\mathfrak{m} \hookrightarrow B/\mathfrak{n} \hookrightarrow B_\mathfrak{n}/\mathfrak{n} \hookrightarrow M_\mathfrak{n}$. So all the statements $A/\mathfrak{m} \hookrightarrow M$, $B/\mathfrak{m} \hookrightarrow M$, $B/\mathfrak{n} \hookrightarrow M$ for some maximal $\mathfrak{n}$, and $B_\mathfrak{n}/\mathfrak{n} \hookrightarrow M_\mathfrak{n}$ are equivalent. But the final one is also equivalent to $\text{depth}_{B_\mathfrak{n}} M_\mathfrak{n} = 0$.

For $d > 0$, there is $f \in \mathfrak{m}$ a non-zero divisor for M, so f is a nonzero divisor for $M_\mathfrak{n}$ for all $\mathfrak{n}$ maximal (because f being a non-zero divisor is detected by $0 \to M \xrightarrow{f} M$ being exact, which implies that $0 \to M_\mathfrak{n} \xrightarrow{f} M_\mathfrak{n}$ is exact since localization is exact). Then since any maximal M-regular sequence has the same length,

$$\text{depth}_A(M/fM) = \text{depth}_A(M) - 1$$
$$\text{depth}_{B_\mathfrak{n}}(M_\mathfrak{n}/fM_\mathfrak{n}) = \text{depth}_{B_\mathfrak{n}}(M_\mathfrak{n}) - 1$$

and we conclude by induction. □

Lemma 19.1.4. *Suppose A is a noetherian local ring and B a finite A-algebra. Let $N \subset M$ be finitely generated B-modules. Then every irreducible component of $\text{supp}_B(N)$ has equidimension $\geq \text{depth}_A(M)$. That is, if $\mathfrak{p} \in \text{spec}(B)$ is the generic point of an irreducible component of $\text{supp}_B(N)$ and if $\mathfrak{n} \supset \mathfrak{p}$ is a maximal ideal, then the length of the maximum chain of ideals from $\mathfrak{n}$ to $\mathfrak{p}$, i.e. $\dim B_\mathfrak{n}/\mathfrak{p}$, satisfies*

$$\dim B_\mathfrak{n}/\mathfrak{p} \geq \text{depth}_A(M).$$

Proof. Choose $\mathfrak{p} \in \operatorname{supp}_B(N)$ minimal, so that we have $B/\mathfrak{p} \hookrightarrow N \hookrightarrow M$. Let $\mathfrak{n} \supset \mathfrak{p}$ be maximal, so that $B_\mathfrak{n}/\mathfrak{p} \hookrightarrow M_\mathfrak{n}$. Then

$$\dim B_\mathfrak{n}/\mathfrak{p} \geq \operatorname{depth}_{B_\mathfrak{n}}(M_\mathfrak{n}) \geq \operatorname{depth}_A(M)$$

by Part 4 of Proposition 19.1.2 and Lemma 19.1.3. □

Lemma 19.1.5. *Suppose A is a noetherian local ring, B a finite A-algebra, and*

$$P^0 \to P^1 \to \cdots \to P^m$$

is a bounded complex of projective A-modules such that the action of A on $H^\bullet(P^\bullet)$ extends to an action of B. Also assume that

$$\dim B + m \leq \operatorname{depth}_A(A).$$

Then

1. $H^i(P^\bullet) = (0)$ *for* $i < m$.
2. $\operatorname{supp}_B(H^m(P^\bullet))$ *is a union of irreducible components of B all of equidimension* $\dim B$.
3. *Every irreducible component of* $\operatorname{supp}_A(H^m(P^\bullet))$ *has dimension equal to* $\dim B$.

In the context of modularity, this was first proven and used by Calegari-Geraghty.

Proof. Choose i minimal such that $H^i(P^\bullet) \neq (0)$. Then $H^i(P^\bullet) \subset P^i/\operatorname{im} d_{i-1}$, and

$$\operatorname{proj\,dim} P^i/\operatorname{im} d_{i-1} \leq i.$$

Let $\mathfrak{p} \in \operatorname{supp}_B(H^i(P^\bullet))$ be minimal and $\mathfrak{n} \supset \mathfrak{p}$ maximal. Then

$$\dim B \geq \dim B_\mathfrak{n}/\mathfrak{p}$$

with equality if and only if $\mathfrak{p}$ is minimal, which by Lemma 19.1.4 (since $B/\mathfrak{p} \hookrightarrow H^i(P^\bullet)$) is

$$\geq \operatorname{depth}_A(P^i/\operatorname{im} d_{i-1})$$

and by Auslander-Buchsbaum this is

$$\geq \operatorname{depth}_A(A) - i \geq \operatorname{depth}_A(A) - m \geq \dim B$$

so we have equalities throughout. So $\dim B = \dim B_\mathfrak{n}/\mathfrak{p}$, implying that $\operatorname{supp}_B(H^i(P^\bullet))$ is a union of irreducible components of $\operatorname{spec} B$ of equidimension $\dim B$ (Part 2), and also $i = m$ (Part 1).

For Part 3, if $\mathfrak{p} \in \operatorname{supp}_A(H^m(P^\bullet))$ is minimal, then

$$\dim B \geq \dim B/\operatorname{ann}_B(M) = \dim A/\operatorname{ann}_A(M)$$

since $A/\operatorname{ann}_A(M) \subset B/\operatorname{ann}_B(M) \subset \operatorname{End}_A(M)$ and the first inclusion is finite, hence the two rings have the same dimension by the going-up theorem. Since $\mathfrak{p}$ contains $\operatorname{ann}_A(M)$ by definition, this is

$$\geq \dim A/\mathfrak{p} \geq \operatorname{depth}_A(H^m(P^\bullet)) = \dim B$$

so $\dim A/\mathfrak{p} = \dim B$. □

In applications, the dimension of the ring A of diamond operators will be significantly larger than the dimension of the ring B of Hecke operators. In particular, A will be a regular ring, so that $\operatorname{depth}_A(A) = \dim A$, which will fortunately actually equal $\dim B + m$, i.e. $\dim B$ plus the length of the projective complex.

19.2 Application to patched modules

We were looking at the surjection $R_{\chi,\infty} \twoheadrightarrow \mathbb{T}_{\chi,\infty}/I_{\chi,\infty}$, where $R_{\chi,\infty}$ is a power series ring over a local lifting ring, $\mathbb{T}_{\chi,\infty}$ is a patched Hecke algebra, and $I^{\delta}_{\chi,\infty} = (0)$. $\mathbb{T}_{\chi,\infty}$ is a subring of $\mathrm{End}_{D(S_\infty)}(C^\bullet_{\chi,\infty})$, a perfect complex of S_∞-modules. We want to prove that $\ker(R_{\chi,\infty} \to \mathbb{T}_{\chi,\infty}/I_{\chi,\infty})$ is nilpotent. We are given that

$$C^\bullet_{\chi,\infty} \otimes^L_{S_\infty} \mathcal{O} \cong \mathscr{C}_{\chi,\emptyset}$$

where $\mathcal{O} = S_\infty/\mathfrak{a}_\infty$, and

$$H^\bullet(\mathscr{C}_{1,\emptyset})[1/l] \neq (0).$$

First we localize at $\mathfrak{a}_\infty$ to get

$$\mathscr{C}^\bullet_{\chi,\infty,\mathfrak{a}_\infty} \otimes^L_{S_{\infty,\mathfrak{a}_\infty}} L \cong \mathscr{C}_{\chi,\emptyset} \otimes_{\mathcal{O}} L.$$

From the discussion in Section 15.1, we know that

$$H^i(\mathscr{C}_{\chi,\emptyset} \otimes_{\mathcal{O}} L) \neq (0)$$

only for $i \in [q_0, q_0 + l_0]$ where

$$q_0 = \frac{1}{2}n(n-1)[F^+ : \mathbb{Q}], \qquad l_0 = [F^+ : \mathbb{Q}]n - 1.$$

Since $S_{\infty,\mathfrak{a}_\infty}$ is a regular local ring, we can find a quasi-isomorphism

$$C^\bullet_{\chi,\infty,\mathfrak{a}_\infty} \cong D^\bullet_{\chi,\infty},$$

where $D^\bullet_{\chi,\infty}$ is a perfect complex of $S_{\infty,\mathfrak{a}_\infty}$-modules concentrated in degrees q_0 to $q_0 + l_0$ (i.e. a minimal representative). Write this out as

$$D^{q_0}_{\chi,\infty} \to D^{q_0+1}_{\chi,\infty} \to \cdots \to D^{q_0+l_0}_{\chi,\infty}.$$

We have

$$\begin{aligned} \dim \mathbb{T}_{\chi,\infty,\mathfrak{a}_\infty} &= \dim \mathbb{T}_{\chi,\infty,\mathfrak{a}_\infty}/I_{\chi,\infty,\mathfrak{a}_\infty} \\ &\leq \dim R_{\chi,\infty,\mathfrak{a}_\infty} \\ &\leq \dim R_{\chi,\infty} - 1 \end{aligned}$$

since $\mathfrak{a}_\infty$ is a proper ideal of $R_{\chi,\infty}$ not containing l, and this is

$$= \dim S_\infty - [F^+ : \mathbb{Q}]n = \dim S_{\infty,\mathfrak{a}_\infty} - l_0$$

($S_{\infty,\mathfrak{a}_\infty}$ being one dimension down from S_∞). So let $B = \mathbb{T}_{\chi,\infty,\mathfrak{a}_\infty}$ and $A = S_{\infty,\mathfrak{a}_\infty}$. We conclude from our lemma that

$$H^i(D^\bullet_{\chi,\infty}) = H^i(C^\bullet_{\chi,\infty,\mathfrak{a}_\infty}) = (0)$$

unless $i = q_0 + l_0 = \frac{1}{2}n(n+1)[F^+ : \mathbb{Q}]$. So before patching, we had cohomology spread out across lots of different degrees, but after patching, we've concentrated the cohomology in one degree. From what we know, we can see that

$$H^{q_0+l_0}(C^\bullet_{1,\infty,\mathfrak{a}_\infty}) \neq (0).$$

Furthermore, every irreducible component of

$$\mathrm{supp}_{\mathbb{T}_{\chi,\infty,\mathfrak{a}_\infty}}\left(H^{q_0+l_0}(C^\bullet_{\chi,\infty,\mathfrak{a}_\infty})\right)$$

has dimension

$$\dim S_\infty - 1 - l_0 = \dim R_{\chi,\infty} - 1 = \dim \mathbb{T}_{\chi,\infty,\mathfrak{a}_\infty}.$$

Therefore, every irreducible component of

$$\mathrm{supp}_{\mathbb{T}_{\chi,\infty}}\left(H^{q_0+l_0}(C^\bullet_{\chi,\infty,\mathfrak{a}_\infty})\right) = \text{closure of } \mathrm{supp}_{\mathbb{T}_{\chi,\infty,\mathfrak{a}_\infty}}\left(H^{q_0+l_0}(C^\bullet_{\chi,\infty,\mathfrak{a}_\infty})\right)$$

has dimension $\geq \dim R_{\chi,\infty}$, since we're no longer inverting l. But since $R_{\chi,\infty} \twoheadrightarrow \mathbb{T}_{\chi,\infty}/I_{\chi,\infty}$, this dimension must in fact equal $\dim R_{\chi,\infty}$. That is,

$$\mathrm{supp}_{\mathbb{T}_{\chi,\infty}}\left(H^{q_0+l_0}(C^\bullet_{\chi,\infty,\mathfrak{a}_\infty})\right) \subset \mathrm{spec}\,\mathbb{T}_{\chi,\infty} = \mathrm{spec}\,\mathbb{T}_{\chi,\infty}/I_{\chi,\infty} \subset \mathrm{spec}\,R_{\chi,\infty}$$

is a union of irreducible components of $\mathrm{spec}\,R_{\chi,\infty}$. We also know that

$$\mathrm{supp}_{\mathbb{T}_{1,\infty}}\left(H^{q_0+l_0}(C^\bullet_{\chi,\infty,\mathfrak{a}_\infty})\right) \neq \emptyset.$$

If $\mathrm{spec}\,R_{\chi,\infty}$ were irreducible, we would conclude that the above inclusions are equalities and be done. But $\mathrm{spec}\,R_{1,\infty}$ is certainly not irreducible. We now want to use the fact that $R_{\chi_0,\infty}$ is irreducible to pass information to $R_{1,\infty}$. The problem is that we are currently in characteristic 0—localizing at $\mathfrak{a}_\infty$ means inverting l—and we can only compare $R_{\chi_0,\infty}$ and $R_{1,\infty}$ mod l. Now we need more commutative algebra.

19.3 Length lemma

Lemma 19.3.1. *Suppose that T is an excellent local ring (which we will not define—it has to satisfy a long list of technical hypotheses which any noetherian ring you come across will probably satisfy, since excellence is preserved under localization, taking finitely generated algebras, etc.). Suppose that $f \in \mathfrak{m}_T$ and $T/(f)$ has Krull dimension 0 (so it's artinian). (In our case, T will be the patched Hecke algebra and f will be l.)*

Note that T has Krull dimension at most 1 (it can also be 0 if f is nilpotent), so all its prime ideals other than $\mathfrak{m}_T$ are minimal, and since T is noetherian, there are only finitely many of them, say $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ (where r could be 0).

Then there exist $a_i \in \mathbb{Z}_{>0}$ for $i = 1, \dots, r$ with the following property. Suppose that M is a finitely generated T-module. Observe that M/fM and $M[f]$ are finite-length T-modules (being finitely generated modules for the artinian ring $T/(f)$) and $M_{\mathfrak{p}_i}$ will be finite length over $T_{\mathfrak{p}_i}$ (being finitely generated over the artinian ring $T_{\mathfrak{p}_i}$). Then

$$\mathrm{length}_T(M/fM) - \mathrm{length}_T(M[f]) = \sum_{i=1}^r a_i\,\mathrm{length}_{T_{\mathfrak{p}_i}}(M_{\mathfrak{p}_i}).$$

This is useful for us because the LHS is computed mod l whereas the RHS is characteristic zero.

Example 19.3.2. Let $T = \mathbb{Z}_l$, $f = l$, M finitely generated over $\mathbb{Z}_l$. Then the claim is that

$$\dim_{\mathbb{Q}_l} M \otimes \mathbb{Q}_l = \dim_{\mathbb{F}_l} M/lM - \dim_{\mathbb{F}_l} M[l].$$

For example if $M = \mathbb{Z}_l$, this is $1 = 1 - 0$, and if $M = \mathbb{Z}_l/l^a\mathbb{Z}_l$ for $a > 0$, it is $0 = 1 - 1$.

The case $T = \mathbb{Z}_l$ is easy to prove in general because by the structure theorem for finitely generated modules over a PID, we only have to consider direct sums of the above two examples.

Start of proof of Lemma 19.3.1. If $0 \to M_1 \to M_2 \to M_3 \to 0$ is short exact and the desired equality holds for two of the terms, then it holds for the third. This is because we get exact sequences

$$0 \to M_{1,\mathfrak{p}_i} \to M_{2,\mathfrak{p}_i} \to M_{3,\mathfrak{p}_i} \to 0$$

and the snake lemma gives

$$0 \to M_1[f] \to M_2[f] \to M_3[f] \to M_1/fM_1 \to M_2/fM_2 \to M_3/fM_3 \to 0$$

and length is additive on exact sequences. Therefore, the equality is true if M has finite length as a T-module, because we can reduce to the case $M = T/\mathfrak{m}_T$, where it is again $0 = 1 - 1$. In particular it is true if $\dim T = 0$. So we can suppose $\dim T = 1$.

Let J be the nilradical of T (set of all nilpotent elements/intersection of all minimal prime ideals). Since T is noetherian, $J^e = (0)$ for some e. By looking at

$$M \supset JM \supset J^2M \supset \cdots \supset J^eM = (0),$$

we reduce to the case where M is a T/J-module (as $J^{i-1}M/J^iM$ is).

Let $\tilde{T}$ be the normalization of T/J, i.e. $\prod_{i=1}^r \widetilde{(T/\mathfrak{p}_i)}$ where $\widetilde{(T/\mathfrak{p}_i)}$ is the integral closure of $T/\mathfrak{p}_i$ in its field of fractions. Because T is excellent, $\tilde{T}$ is a finitely generated T-module (it is not true for all noetherian rings, or even noetherian domains, that the integral closure is finitely generated). Since $T/J \hookrightarrow \prod T/\mathfrak{p}_i \hookrightarrow \prod \widetilde{(T/\mathfrak{p}_i)}$, we get a short exact sequence

$$0 \to T/J \to \tilde{T} \to Q \to 0$$

where Q has trivial localization at each $\mathfrak{p}_i$, that is, $\operatorname{supp} Q \subset \{\mathfrak{m}_T\}$. So Q has finite length.

Let M be a finitely generated T/J-module. We get

$$\operatorname{Tor}_1^T(M, Q) \to M \to M \otimes \tilde{T} \to M \otimes Q \to 0$$

and since tensor products and Tor commute with localization, $\operatorname{Tor}_1^T(M, Q)$ and $M \otimes Q$ have support in $\{\mathfrak{m}_T\}$ and are finite length over T, so we already know the lemma for them. We conclude that it suffices to prove the desired equality for M a finitely generated $\tilde{T}$-module, or even a finitely generated $\widetilde{(T/\mathfrak{p}_i)}$-module.

So suppose M is a finitely generated $\widetilde{(T/\mathfrak{p}_i)}$-module. If $j \neq i$, $M_{\mathfrak{p}_j} = (0)$, since $(T/\mathfrak{p}_i)_{\mathfrak{p}_j} = 0$. So we have to find $a_i \in \mathbb{Z}_{>0}$ such that for any finitely generated $\widetilde{(T/\mathfrak{p}_i)}$-module M, we have

$$\text{length}_T(M/fM) - \text{length}_T(M[f]) = a_i \, \text{length}_{T_{\mathfrak{p}_i}}(M_{\mathfrak{p}_i}).$$

Note that $\widetilde{(T/\mathfrak{p}_i)}$ is an integrally closed domain; being dimension 1 and noetherian, it's a Dedekind domain. It will turn out that $\widetilde{(T/\mathfrak{p}_i)}$ furthermore has finitely many maximal ideals, so is a PID, hence we can use the same argument that we used for $\mathbb{Z}_l$. We will finish this next time. □

Next time, we will explain a derived version of this lemma and then finish the argument.

20 June 3: proof of main theorem.

20.1 Lemma from last time

Remark 4. The following are properties of excellent rings.

- Excellent implies noetherian, and any complete local noetherian ring is excellent.
- Excellence is preserved under localization and quotients.
- A finitely generated algebra (as an algebra, it doesn't have to be as a module) over an excellent ring is excellent.
- Fields and Dedekind domains are excellent.

Lemma 20.1.1. *Suppose T is an excellent local ring. Suppose that $f \in \mathfrak{m}_T$ and $T/(f)$ has Krull dimension 0 (hence is artinian, because it's noetherian). Then T has finitely many prime ideals other than $\mathfrak{m}_T$, say $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$, and these are all minimal primes. If M is a finitely generated T-module, then*

$$\text{length}_T(M/fM) - \text{length}_T(M[f]) = \sum a_i \, \text{length}_{T_{\mathfrak{p}_i}}(M_{\mathfrak{p}_i})$$

where the $a_i \in \mathbb{Z}_{>0}$ are independent of M.

Finishing the proof of Lemma 20.1.1. Let $\widetilde{T/\mathfrak{p}_i}$ be the integral closure of $T/\mathfrak{p}_i$ in its field of fractions. Because T is excellent, $\widetilde{T/\mathfrak{p}_i}$ is finitely generated as a $T/\mathfrak{p}_i$-module. We had reduced to the case that M is a $\widetilde{T/\mathfrak{p}_i}$-module, in which case we need to prove just that

$$\text{length}_T(M/fM) - \text{length}_T(M[f]) = a_i \, \text{length}_{T_{\mathfrak{p}_i}}(M_{\mathfrak{p}_i}).$$

Now, $T/\mathfrak{p}_i$ is a 1-dimensional noetherian domain, so $\widetilde{T/\mathfrak{p}_i}$ is a 1-dimensional noetherian integrally closed domain, i.e. a Dedekind domain. Also $T/\mathfrak{p}_i$ is local and has a unique maximal ideal, so $\widetilde{T/\mathfrak{p}_i}$ has only finitely many maximal ideals above it by the going-up theorem. This actually implies that $\widetilde{T/\mathfrak{p}_i}$ is a PID. This is because ideals of Dedekind domains factor as products of powers of prime ideals, so you just have to check that the maximal ideals are principally generated. You can find generators of a given maximal ideal $\mathfrak{m}$ using the Chinese remainder theorem: you choose $a \in \mathfrak{m} \setminus \mathfrak{m}^2$, then choose b such that $b \equiv a \pmod{\mathfrak{m}^2}$ and $b \equiv 1 \pmod{\mathfrak{m}'}$ for $\mathfrak{m}' \neq \mathfrak{m}$, and it turns out that b generates $\mathfrak{m}$.

So M is a direct sum of a finite-length module and $\left(\widetilde{T/\mathfrak{p}_i}\right)^{\oplus d}$ for some d. We already know the lemma for finite-length modules, and that in short exact sequences if it is true for two terms then it is true for the third, so we only need to prove it for $\widetilde{T/\mathfrak{p}_i}$ itself. So it suffices to find $a_i \in \mathbb{Z}_{>0}$ such that

$$\operatorname{length}_T\left(\widetilde{T/\mathfrak{p}_i}/(f)\right) - \operatorname{length}_T\left(\widetilde{T/\mathfrak{p}_i}[f]\right) = a_i \operatorname{length}_{T_{\mathfrak{p}_i}}\left(\widetilde{T/\mathfrak{p}_i}\right)_{\mathfrak{p}_i} = a_i$$

(since $\left(\widetilde{T/\mathfrak{p}_i}\right)_{\mathfrak{p}_i}$ is just the residue field at $\mathfrak{p}_i$). But $\widetilde{T/\mathfrak{p}_i}$ is a domain, so it has no f-torsion, and the second term on the LHS is 0. So we just take

$$a_i = \operatorname{length}_T\left(\widetilde{T/\mathfrak{p}_i}/(f)\right) > 0$$

which is positive because f is not a unit in $\widetilde{T/\mathfrak{p}_i}$ (because it is in $\mathfrak{m}_T$, hence in every maximal ideal of $\widetilde{T/\mathfrak{p}_i}$, since they all lie over $\mathfrak{m}_T$). □

20.2 Derived version

Let S be a ring, $C \in D^b(S)$, and T an S-algebra such that $T \hookrightarrow \operatorname{End}_{D^b(S)}(C)$. If $H^i(C)$ has finite length over T for all i, define

$$\operatorname{length}_T(C) = \sum_i (-1)^i \operatorname{length}_T H^i(C).$$

If $C_1 \to C_2 \to C_3 \to$ is an exact triangle with each $H^i(C_j)$ finite length over T, then

$$\operatorname{length}_T(C_2) = \operatorname{length}_T(C_1) + \operatorname{length}_T(C_3).$$

Actually if for any two values of j we have that $H^i(C_j)$ is finite length over T for all i, then also $H^i(C_j)$ is finite length over T for all j, i and the above formula holds.

Lemma 20.2.1. *Suppose S is an excellent local ring and $f \in \mathfrak{m}_S$ is a non-zero divisor. Suppose T is a finite S-algebra (finitely generated as a module) such that T/fT has Krull dimension 0. Then $T/(f) = \prod_{\mathfrak{n}}(T/(f))_{\mathfrak{n}}$ as $\mathfrak{n}$ runs over maximal ideals of $T/(f)$ (of which there are finitely many). Let $e_{\mathfrak{n}}$ denote the idempotent $1 \in (T/(f))_{\mathfrak{n}}$. Choose one such $\mathfrak{n}$, and call it $\mathfrak{m}$.*

Then $T_{\mathfrak{m}}$ has finitely many primes other than $\mathfrak{m}$, all minimal primes; call them $\mathfrak{x}_1, \ldots, \mathfrak{x}_r$. Write $\mathfrak{y}_i$ for the pullback of $\mathfrak{x}_i$ to S, a prime ideal in S. Then $T_{\mathfrak{y}_i}$ is artinian and

$$T_{\mathfrak{y}_i} = \prod_{j \mid \mathfrak{y}_j = \mathfrak{y}_i} T_{\mathfrak{x}_j}.$$

Let $e_{\mathfrak{x}_j}$ be the idempotent in $T_{\mathfrak{y}_i}$ corresponding to $1 \in T_{\mathfrak{x}_j}$.

Then there exist $a_j \in \mathbb{Z}_{>0}$ (for j such that $\mathfrak{y}_j = \mathfrak{y}_i$) so that the following holds. If $C \in D^b(S)$ such that C and $C \otimes_S^L S/(f)$ have finitely generated cohomology (over S, or T for that matter) and if $T \to \operatorname{End}_{D^b(S)}(C)$, then

$$\operatorname{length}_{T_{\mathfrak{m}}}\left(e_{\mathfrak{m}} C \otimes_S^L S/(f)\right) = \sum_i a_i \operatorname{length}_{T_{\mathfrak{x}_i}}(e_{\mathfrak{x}_i}(C \otimes_S S_{\mathfrak{y}_i}))$$

(remember $e_{\mathfrak{x}_i} \in T_{\mathfrak{y}_i}$). This makes sense because the cohomology of $e_{\mathfrak{m}} C \otimes_S^L S/(f)$ and $e_{\mathfrak{x}_j} C \otimes_S S_{\mathfrak{y}_i}$ have finite length over $T_{\mathfrak{m}}$, $T_{\mathfrak{x}_j}$, because the cohomology is finitely generated and the rings are artinian.

Proof. Take a_i as in the previous lemma for $T_{\mathfrak{m}}$. Argue by induction on the range (maximum minus minimum) of $\{i \mid H^i(C) \neq (0)\}$.

First suppose the range is ≤ 0 (we write $\leq$ because it's conceivable that the thing has no cohomology). Then there is i such that $H^j(C) = (0)$ for $j \neq i$. Then $\tau_{\leq i} C \to C$ is a quasi-isomorphism. Furthermore $\tau_{\leq i} C \to H^i(C)[-i]$ is a quasi-isomorphism. So WLOG $C = M[-i]$ for some module M finitely generated over S. Look at

$$M[-i] \otimes_S^L S/(f) \cong M[-i] \otimes_S^L \left(S[\text{in degree } -1] \xrightarrow{f} S[\text{in degree } 0]\right)$$

(since f is not a zero divisor in S, this complex works—it only has cohomology in degree 0, where it is $S/(f)$). These are flat, so we get the complex

$$M[\text{in degree } i-1] \xrightarrow{f} M[\text{in degree } i]$$

so we have

$$H^j\left(M[-i] \otimes_S^L S/(f)\right) = \begin{cases} M[f] & \text{if } j = i-1 \\ M/fM & \text{if } j = i \\ (0) & \text{otherwise.} \end{cases}$$

Therefore

$$\operatorname{length}_{T_{\mathfrak{m}}}\left(e_{\mathfrak{m}} M[-i] \otimes_S^L S/(f)\right) = (-1)^i \left(\operatorname{length}_{T_{\mathfrak{m}}}(e_{\mathfrak{m}} M/fM) - \operatorname{length}_{T_{\mathfrak{m}}}(e_{\mathfrak{m}} M[f])\right).$$

On the other hand,

$$\operatorname{length}_{T_{\mathfrak{x}_i}}(e_{\mathfrak{x}_i}(M[-i] \otimes_S S_{\mathfrak{y}_i})) = (-1)^i \operatorname{length}_{T_{\mathfrak{x}_i}}(M_{\mathfrak{y}_i})_{\mathfrak{x}_i} = (-1)^i \operatorname{length}_{T_{\mathfrak{x}_i}}(M_{\mathfrak{x}_i})$$

since everything is already flat, and we reduce to the previous lemma.

Now if the range is > 0, choose i maximal such that $H^i(C) \neq (0)$. We have an exact triangle

$$\tau_{<i} C \to C \to H^i(C)[-i] \to .$$

The range of $\tau_{<i} C$ is smaller than the range of C and the range of $H^i(C)[-i]$ is 0, so in order to induct, we only need to check that $H^j(\tau_{<i} C)$ and $H^j\left(\tau_{<i} C \otimes_S^L S/(f)\right)$ are finitely

generated over T for all j. The first is clear because it's the same as $H^j(C)$ for $j < i$ and 0 otherwise. For the second, we have an exact triangle

$$\tau_{<i}C \otimes_S^L S/(f) \to C \otimes_S^L S/(f) \to H^i(C)[-i] \otimes_S^L S/(f) \to .$$

Now again $H^i(C) \otimes_S^L S/(f)[-i]$ is represented by

$$H^i(C)[\text{in degree } i-1] \xrightarrow{f} H^i(C)[\text{in degree } i].$$

So it only has cohomology in degrees $i-1$ and i, and we have

$$H^j\left(\tau_{<i}C \otimes_S^L S/(f)\right) \xrightarrow{\sim} H^j\left(C \otimes_S^L S/(f)\right)$$

if $j \leq i-2$, and for $j = i-1$ or $j = i$, we have the long exact sequence

$$\begin{aligned} 0 &\to H^{i-1}\left(\tau_{<i}C \otimes_S^L S/(f)\right) \to H^{i-1}\left(C \otimes_S^L S/(f)\right) \to H^i(C)[f] \\ &\to H^i\left(\tau_{<i}C \otimes_S^L S/(f)\right) \to H^i\left(C \otimes_S^L S/(f)\right) \to H^i(C)/(f) \\ &\to H^{i+1}\left(\tau_{<i}C \otimes_S^L S/(f)\right) \to 0 \end{aligned}$$

and because $H^{i-1}\left(C \otimes_S^L S/(f)\right)$, $H^i(C)[f]$, $H^i\left(C \otimes_S^L S/(f)\right)$, and $H^i(C)/(f)$ are finite-length, so are the others.

So we can apply the inductive hypothesis to $\tau_{<i}C$ and the case of range 0 to $H^i(C)$, and we conclude. ☐

20.3 Main theorem

Proposition 20.3.1. *$R_{1,\infty} \twoheadrightarrow \mathbb{T}_{1,\infty}/I_{1,\infty}$ has nilpotent kernel.*

Proof. We have a closed embedding $\operatorname{spec} \mathbb{T}_{\chi,\infty} \subset \operatorname{spec} R_{\chi,\infty}$. Let x_1 be a generic point of $\mathbb{T}_{1,\infty,\mathfrak{a}_\infty}$. Then by the argument from the previous lecture, it is also a generic point of $R_{1,\infty}$ (since we saw that $\operatorname{spec} \mathbb{T}_{\chi,\infty}$ was a union of irreducible components of $\operatorname{spec} R_{\chi,\infty}$). Let x_2 be any generic point of $R_{1,\infty}$. It suffices to prove that $x_2 \in \operatorname{spec} \mathbb{T}_{1,\infty}$.

Let $\overline{x}_1, \overline{x}_2$ be generic points of $R_{1,\infty}/\lambda$ specializing to x_1, x_2 respectively. Then $\overline{x}_1 \in \operatorname{spec} \mathbb{T}_{1,\infty}$. Furthermore, x_1, x_2 are the unique generic points of $R_{1,\infty}$ specializing to $\overline{x}_1, \overline{x}_2$.

Let $y_1, y_2, \overline{y}_1, \overline{y}_2$ be the contractions of $x_1, x_2, \overline{x}_1, \overline{x}_2$ to S_∞. Then $y_1 \subset \mathfrak{a}_\infty$. Let $\overline{x}'_1, \overline{x}'_2$ be the primes of $R_{\chi 0,\infty}$ corresponding to $\overline{x}_1, \overline{x}_2$ under the isomorphism $R_{\chi 0,\infty}/\lambda \cong R_{1,\infty}/\lambda$. Let x' be the unique generic point of $R_{\chi 0,\infty}$ and y' the contraction of x' to S_∞.

By flatness, since $y_1 \subset \mathfrak{a}_\infty$, we have

$$H^\bullet(C_{1,\infty,y_1}) \cong H^\bullet(C_{1,\infty,\mathfrak{a}_\infty})_{y_1} = \begin{cases} 0 & \text{if } \bullet \neq q_0 + l_0 \\ \text{not } 0 & \text{if } \bullet = q_0 + l_0. \end{cases}$$

Now let $S = S_{\infty,\overline{y}_1}$, $T = \mathbb{T}_{1,\infty,\overline{y}_1}$, $C = C_{1,\infty,\overline{y}_1}$, $f = l$, $\mathfrak{m} = \overline{x}_1$, and apply Lemma 20.1.1 for the first of four times. We have

$$\dim T/l = \dim \mathbb{T}_{1,\infty}/l - \dim \mathbb{T}_{1,\infty}/\overline{y}_1 = \dim \mathbb{T}_{1,\infty}/l - \dim S_\infty/\overline{y}_1$$

by the going-up theorem, since $\mathbb{T}_{1,\infty}/\overline{y}_1$ contains and is finite over $S_\infty/\overline{y}_1$. We know that

$$\dim \mathbb{T}_{1,\infty}/l = \dim R_{1,\infty} - 1$$

since $\mathbb{T}_{1,\infty}$ contains a component of $R_{1,\infty}$. But also, again by the going-up theorem,

$$\dim S_\infty/\overline{y}_1 = \dim \mathbb{T}_{1,\infty}/\overline{x}_1 = \dim R_{1,\infty}/\overline{x}_1 = \dim R_{1,\infty} - 1$$

so $\dim T/l = 0$.

Now, we have

$$\operatorname{spec} \mathbb{T}_{1,\infty,\overline{x}_1} = \operatorname{spec} R_{1,\infty,\overline{x}_1} = \{x_1, \overline{x}_1\}.$$

By assumption,

$$\text{length}_{S_{\infty,y_1}}(e_{x_1} C_{1,\infty} \otimes_{S_\infty} S_{\infty,y_1}) \neq (0)$$

because the cohomology is nonzero in exactly one degree (so that the alternating sum doesn't cancel anything out), so Lemma 20.1.1 implies

$$\text{length}_{S_{\infty,y_1}} \left(e_{\overline{x}_1} C_{1,\infty} \otimes^L_{S_\infty} S_{\infty,\overline{y}_1}/l\right) \neq 0.$$

But also

$$\text{length}_{S_{\infty,y_1}} \left(e_{\overline{x}_1} C_{1,\infty} \otimes^L_{S_\infty} S_{\infty,\overline{y}_1}/l\right) = \text{length}_{S_{\infty,\overline{y}_1}} \left(e_{\overline{x}'_1} C_{\chi_0,\infty} \otimes^L_{S_\infty} S_{\infty,\overline{y}'_1}/l\right).$$

Apply the lemma again with $S = S_{\infty,\overline{y}_1}$, $T = \mathbb{T}_{\chi_0,\infty,\overline{y}_1}$, $C = C_{\chi_0,\infty} \otimes_{S_\infty} S_{\infty,\overline{y}_1}$, $f = l$, $\mathfrak{m} = \overline{x}'_1$. As before, $\dim T/l = 0$. Because the RHS of the above equality is nonzero, we conclude that $\dim \mathbb{T}_{\chi_0,\infty,\overline{x}'_1} = 1$, so

$$\operatorname{spec} \mathbb{T}_{\chi_0,\infty,\overline{x}'_1} = \{\overline{x}'_1, x'\} = \operatorname{spec} R_{\chi,\infty,\overline{x}'_1}.$$

because x' is the unique generic point of $R_{\chi_0,\infty}$. Since $\operatorname{spec} \mathbb{T}_{\chi_0,\infty}$ is closed in $R_{\chi_0,\infty}$ and contains its unique generic point, we have

$$\operatorname{spec} \mathbb{T}_{\chi_0,\infty} = \operatorname{spec} R_{\chi_0,\infty}.$$

This is what we wanted for χ_0, not 1. Now we go back. We know that

$$\text{length}_{S_{\infty,y'}}(C_{\chi_0,\infty} \otimes_{S_\infty} S_{\infty,y'}) \neq 0.$$

Apply the lemma again with $S = S_{\infty,\overline{y}_2}$, $T = \mathbb{T}_{\chi_0,\infty,\overline{y}_2}$, $C = C_{\chi_0,\infty} \otimes_{S_\infty} S_{\infty,\overline{y}_2}$, $f = l$, $\mathfrak{m} = \overline{x}'_2$. As before $\dim T/l = 0$, so back in characteristic l we have

$$\text{length}_{S_{\infty,\overline{y}_2}}(e_{\overline{x}_2} C_{\chi_0,\infty} \otimes_{S_\infty} S_{\infty,\overline{y}_2}) \neq 0.$$

This is the same as

$$\text{length}_{S_{\infty,\overline{y}_2}}(e_{\overline{x}_2} C_{1,\infty} \otimes_{S_\infty} S_{\infty,\overline{y}_2})$$

and now we apply the lemma a final time with $S = S_{\infty,\overline{y}_2}$, $T = \mathbb{T}_{1,\infty,\overline{y}_2}$, $C = C_{1,\infty} \otimes_{S_\infty} S_{\infty,\overline{y}_2}$, $f = l$, $\mathfrak{m} = \overline{x}_2$. Since the length written above is nonzero, again $\dim T/l = 0$. T has at most one prime contained in $\overline{x}_2$, so $\dim T = 1$ and in fact $\dim T_{\overline{x}_2} = 1$, i.e. $x_2 \in \operatorname{spec} \mathbb{T}_{1,\infty,\overline{x}_2}$, hence $x_2 \in \operatorname{spec} \mathbb{T}_{1,\infty}$, as desired. □

In summary, we start with x_1 being automorphic; by applying the lemma twice, we get that the unique generic point x' of $R_{\chi_0,\infty}$ is automorphic; by coming back to $R_{1,\infty}$ via $\overline{y}_2$, we get that x_2 is automorphic, hence spreading the automorphy to the second component.

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